R.L. STRATONOVICH

TOPICS IN THE THEORY OF RANDOM NOISE

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Peaks of Random Functions and the Effect of Noise on Relays Nonlinear Self-Excited Oscillations in the Presence of Noise

Author's Preface to Volumes I and II

Random processes are of great and ever increasing interest to scientists and engineers working in various branches of radio physics. This is quite understandable, since at the present stage of technological development, random noise constitutes the chief obstacle to further improvement of certain engineering devices. In fact, a current problem in many fields of radio physics is that of evolving techniques for analysis and design of equipment which is immune to noise. Convincing proof of the importance and urgency of this subject is afforded by the large number of books devoted to it, published both in the U.S.S.R. and elsewhere.

The present book, in two volumes, is concerned with a variety of topics which have been left to one side, as it were, but which are nevertheless of great practical importance. In particular, we have in mind problems involving the effects of noise on electronic relays and vacuum-tube oscillators. Recently, there has been increased interest in the operation of switching devices and self-excited systems, not only because of the development of methods of coherent radar detection, but also in connection with such problems as increasing the accuracy of time measurements, designing particle accelerators and computing machines, etc. However, the journal literature in these new fields of statistical radio engineering is hard to penetrate. In the present book, noise in relays and oscillators is studied in an organized fashion, from a rather general point of view. Material that can be found in the literature has been systematized, and the results of some original investigations by the author have been included. The physical treatment of the results obtained may be a bit sketchy, but it seems to us that brevity coupled with wide coverage is justified in this instance, since the rapid growth in the total number of published papers has increased the need for some kind of survey, even one resembling a handbook.

From a mathematical point of view, the topics studied here involve nonlinear transformations of random functions, mostly transformations with memory. Certain mathematical methods are worked out in a form suitable for application to problems of radio engineering, and are then exploited systematically. As the reader will discover, the following methods turn out to be effective:

1. Reduction of a given process to a process without aftereffect, followed by application of Markov process techniques, in particular, the Fokker-Planck equation;
2. Linearization of the original equations, which allows us to apply correlation theory in the linear approximation;
3. The quasi-static approach, which allows us to reduce a nonlinear transformation with memory to one without memory.

The treatments in the available literature usually deal only with the correlation theory, or only with the theory of Markov processes. However, to solve concrete problems, one must be able to go from the methods of either theory to those of the other, choosing the best methods for solving the problem at hand. For this reason, we pay special attention to the problem of when, and in what sense, the theory of Markov processes can be applied to actual fluctuations, which are originally described in terms of the correlation theory.

Part 1 of Volume I is devoted to a review of the mathematical results used later in the book. In many places, it has seemed appropriate, for the sake of brevity, to depart from the usual presentation of the basic concepts and results of probability theory. The reader who is not familiar with this necessary background material can refer to any one of a number of textbooks. Those already acquainted with elementary probability theory will find that Part 1 contains a different treatment of certain well-known topics, in addition to some new results. In fact, Part 1 ought not to be devoid of interest even to mathematicians, provided, of course, that they make due allowances for the heuristic level of rigor adopted.

In Volume I, Part 2 and in Volume II, we systematically use the methods just enumerated to solve specific problems of statistical radio engineering. At the same time, we try to explain the conditions under

which the various methods of analyzing noise phenomena are applicable. Perhaps some of the methods are given in more detail than is necessary for immediate applications, but this is done with a view to further development of the tools needed for future work. It is reasonable to expect that the progressively more complicated problems which come up in engineering practice will require the use of progressively more sophisticated methods of analysis. The author will regard his task as accomplished if this book serves in some measure to broaden the theoretical outlook of radio scientists, by helping them master certain methods which they may find novel at first. Although the material given here is intended primarily for specialists in the field of statistical radio engineering, it should also interest scientists working in other fields where similar statistical methods can be profitably used.

As a rule, no attempt is made to cite the papers where various results appear for the first time. This is because any attempt to make systematic source references would be likely to entail errors, unless one wishes to become involved in special bibliographic research.

Finally, the author would like to take this opportunity to thank V. I. Tikhonov, who suggested that the present book be written and rendered substantial assistance, Y. L. Klimontovich, who in his capacity as editor made many valuable suggestions while the manuscript was being prepared for the printer, and I. G. Akopyan, who constructed the figures appearing in Volume II, Chapter 9. The author would also like to thank S. P. Strelkov, S. D. Gvozdover, B. R. Levin and S. A. Akhmanov for their helpful advice, as well as Y. M. Romanovski, P. S. Landa and V. N. Ivanov for their assistance with the manuscript. He also expresses his gratitude to Dr. R. A. Silverman for undertaking to translate and edit the English-language edition of this book, thereby greatly enlarging its prospective audience.

R. L. S.

Translator's Preface to Volume II

As in the first volume, I have worked through all the mathematical derivations in detail. However, since the subject matter of the second volume (noise in relays and oscillators) is more difficult than that of the first, a stronger interaction with the author was necessary, taking the form of an extended correspondence. During this period of almost a year, Dr. Stratonovich has examined the translation in manuscript, batch by batch, sparing no pains to answer all my queries, however persistent, with great wit and wisdom. This interplay has led to betterments of all kinds, in particular the supplements to Chapters 2, 4, 5 and 9. Moreover, it has had the pleasant side effect of establishing a friendship which I hope the future will afford occasion to renew.

R. A. S.

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PART 1

Peaks of Random Functions and the Effect of Noise on Relays

CHAPTER 1

The Mean Number of Peaks of a Random Function

In practice, one encounters a great variety of nonlinear transformations of random functions. Typically, a nonlinear device acts upon an input signal consisting of a useful signal contaminated by noise. Some transformations are only slightly nonlinear, while others are characteristically nonlinear. Among the latter we cite systems (e.g. various kinds of relays with discontinuous transfer characteristics) whose response undergoes a jump discontinuity depending on the size of the input signal. It is difficult to give a general theoretical analysis of nonlinear systems. However, separate categories of nonlinear systems can be studied by special methods, applicable in restricted regions.

This and the following two chapters are devoted to the study of some methods suitable for analysing nonlinear devices of the relay type, which respond when the input signal $x(t)$ exceeds a certain fixed level b. If at the time t the variable $x(t)$ "up-crosses" the level b, i.e., reaches and immediately thereafter exceeds b, then the function $x(t)$ is said to have a "peak" at time t. Sometimes relay-type systems are used to count peaks. For example, it might be required to determine the number of input pulses in a system where each pulse produces its own peak. The presence of noise in the input signal, along with the useful pulses, can lead to a "false peak," i.e., a peak entirely due to the noise rather than to the useful signal, which however causes the relay to operate. (This might be called "false operation" of the relay.) Thus

the presence of noise introduces an error in the number of counted peaks, and a theoretical problem posed by practical considerations is to determine the size of this error.

The theoretical problem is often complicated by the fact that a relay cannot usually respond to each peak, thereby counting the exact number of peaks. In fact, if the peaks follow one another too closely, some of them will be lost because of the device's insufficient resolving power. Even a single peak can be lost if it is too short or too weak. Thus, instead of just calculating the exact number of peaks, the theory should give a more detailed description of the properties of the peaks such as the distribution of duration and power of the peaks.

From the above remarks it is evident that the theory of operation of nonlinear systems of the relay type intimately involves the study of peaks of random functions and their properties. The results so obtained turn out to be useful in many other contexts (e.g., problems involving durability and fatigue of materials).

1. The Mean Number of Peaks of a Smoothly Varying Process

Suppose we are given a random process $x(t)$ such that $x(t)$ is close to the line segment $x(t') \approx x(t)+\dot{x}(t)(t'-t)$ during a sufficiently small time interval $t \leqslant t' \leqslant t+\Delta t$. (The significance of this *smoothness condition* will soon be apparent.) Then the points at which $x(t)$ crosses the level $x=b$ cannot lie too close together, and in fact no more than one such point of intersection can occur in the indicated interval. To calculate the mean number of peaks which begin in the interval from t to $t+\Delta t$, let P_1 denote the probability that one such peak occurs and let P_0 denote the probability that no such peak occurs. Obviously, there are no other possibilities, and the mean number of peaks beginning in the interval is just P_1. It will be assumed that we know the joint two-dimensional probability density $w(x, \dot{x})$ of $x(t)$ and its derivative $\dot{x}(t)$. Then the expression

$$dP = w(x, \dot{x})\Delta x \Delta \dot{x} \tag{1.1}$$

is the probability that the approximately linear curve $x(t)$ intersects

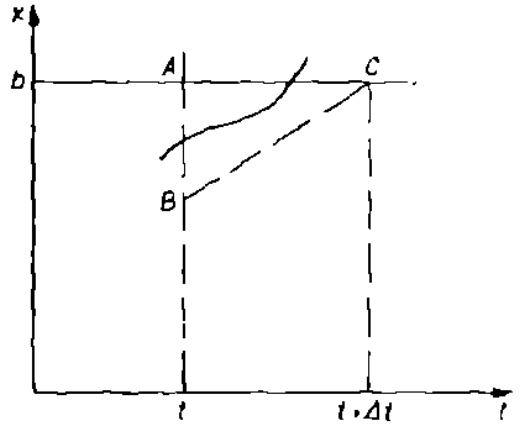

Fig. 1. Beginning of a peak during the interval $[t, t+\Delta t]$.

the vertical segment AB of length Δx shown in Figure 1, while at the same time its derivative lies between $\dot{x}(t)$ and $\dot{x}(t)+\Delta\dot{x}$.

Now consider the probability of an intersection not with the vertical segment AB but with the horizontal segment AC of length Δt. If the derivative $\dot{x}(t)$ is fixed, then crossing a horizontal segment of length Δt is equivalent to crossing a vertical segment of length $\Delta x=\dot{x}\Delta t$. Therefore the probability of crossing the segment AC (which corresponds to the values $t\leqslant t'\leqslant t+\Delta t$, $x=b$) and having a derivative between $\dot{x}$ and $\dot{x}+\Delta\dot{x}$ equals

$$dP = w(x,\dot{x})\dot{x}\Delta t\Delta\dot{x} \quad (x=b). \tag{1.2}$$

It follows on the one hand that the total probability of crossing AB is

$$P_1 = \Delta t\int_0^\infty w(b,\dot{x})\,\dot{x}\,d\dot{x}, \tag{1.3}$$

and on the other hand that the probability density of the derivative $\dot{x}$ at the moment of inception of a peak is

$$w_i(\dot{x}) = \begin{cases} \dfrac{w(b,\dot{x})\,\dot{x}}{\displaystyle\int_0^\infty w(b,\dot{x})\,\dot{x}\,d\dot{x}} & \text{for} \quad \dot{x}>0, \\ 0 & \text{for} \quad \dot{x}<0. \end{cases} \tag{1.4}$$

As already indicated, the probability (1.3) is just the mean number of peaks which begin in the interval $[t, t+\Delta t]$. Dividing P_1 by Δt, we find that the mean density of peaks, i.e., the density of their initial times, is given by

$$n_0(t) = \int_0^\infty w(b, \dot{x})\, \dot{x}\, d\dot{x}. \tag{1.5}$$

For a stationary process $x(t)$, the density (1.5) is independent of t, and is just the mean number of peaks occurring per unit time. To obtain the total number of peaks in the interval $[t_0, t_1]$ for the case of a non-stationary process, we have to evaluate the integral

$$\int_{t_0}^{t_1} n_0(t)\, dt.$$

In what follows, we shall be concerned mainly with stationary processes.

Example. Suppose $x(t)$ is a stationary Gaussian process with mean value zero and correlation function $\langle x x_\tau \rangle = \sigma^2 R(\tau)$. Because of the evenness of $R(\tau)$, there is no correlation between $x(t)$ and its derivative $\dot{x}(t)$ at the same time, i.e.,

$$\langle x\dot{x} \rangle = \frac{d}{d\tau} \langle x x_\tau \rangle \Big|_{\tau=0} = 0. \tag{1.6}$$

For a Gaussian process, lack of correlation between x and $\dot{x}$ implies that x and $\dot{x}$ are statistically independent. Therefore $w(x, \dot{x})$ factors into a product of one-dimensional probability densities, so that formulas (1.4) and (1.5) become

$$w_1(\dot{x}) = \frac{w(\dot{x})\, \dot{x}}{\int_0^\infty w(\dot{x})\, \dot{x}\, d\dot{x}} = \frac{2}{\langle |\dot{x}| \rangle}\, w(\dot{x})\, \dot{x} \quad (\dot{x} > 0), \tag{1.7}$$

and

$$n_0 = \frac{w(b)}{2} \langle |\dot{x}| \rangle.$$

Using the expression (I.3.6) for a Gaussian probability density,[1] and bearing in mind that

$$\langle \dot{x} \rangle = 0, \quad \sigma(\dot{x}) = \sqrt{R_2},$$

where

$$R_2 = -\left.\frac{d^2R(\tau)}{d\tau^2}\right|_{\tau=0},$$

we find that

$$w_1(\dot{x}) = \frac{\dot{x}}{\sigma^2 R_2} e^{-\dot{x}^2/2\sigma^2 R_2} \quad (\dot{x} > 0) \tag{1.8}$$

and

$$n_0 = \frac{\sqrt{R_2}}{2} e^{-b^2/2\sigma^2}. \tag{1.9}$$

Thus we see that the derivative $\dot{x}$ at the time of inception of a peak has a Rayleigh distribution.

The smoothness condition imposed on the function $x(t)$ at the beginning of this section was used when we assumed that $x(t)$ is differentiable and close to a line segment during a sufficiently small time interval. In the present case, this is tantamount to the assumption that $\langle x x_\tau \rangle$ has a finite second derivative at $\tau = 0$.

2. The Mean Number of Peak Clusters of a Markov Process

In all cases actually encountered in radio engineering practice, the process $x(t)$ certainly satisfies the smoothness condition discussed in the preceding section. In fact, the inductances and capacitances always present in the circuitry (whether wanted or not) act to smooth electrical

[1] The book by R. L. Stratonovich, *Topics in the Theory of Random Noise, Volume I*, revised English edition, translated by R. A. Silverman, Gordon and Breach Science Publishers, New York (1963), will henceforth be referred to simply as *Volume I*, and equation numbers in this reference will be preceded by the Roman numeral I. Thus (I.3.6) means equation (3.6) of Volume I, and so on. Similarly, Chapter I.4 means Chapter 4 of Volume I, p. I.81 means page 81 of Volume I, and so on.

signals, if only with a small time constant τ_{sm}. Correspondingly, the spectral density of the signal falls off rapidly at frequencies exceeding an *upper cutoff frequency* $\omega_c \sim 1/\tau_c$ (see Chap. I.4, Secs. 2 and 12). In situations where the smoothing time τ_{sm} is sufficiently small (i.e., smaller than the other important time constants), instead of explicitly introducing the smoothing time τ_{sm}, we shall find it convenient to replace the actual process by a "neighboring" unsmoothed process, like the processes without aftereffect or Markov processes, studied in Vol. I.

Markov processes are a convenient abstraction and are in many respects simpler than smoothed (analytic) processes. However, the considerations given in the preceding section do not apply to Markov processes, since they are everywhere nondifferentiable, as noted in Chap. I.4. In fact, if $x(t)$ is Markovian, the derivative $dx(t)/dt$ has infinite variance and continually changes sign, chaotically oscillating up and down with "infinite frequency" (cf. p.I.81). Therefore, if $x(t)$ intersects the level b at time t, there are an infinite number of other intersections at closely neighboring times, which means that the mean number of peaks is infinite.

To illustrate this point, consider an exponentially correlated process $x(t)$ which has been subjected to the smoothing operation (I.4.280) with time constant τ_{sm}. According to (I.4.284), the smoothed process $y(t)$ has the correlation coefficient

$$R_y(\tau) = \frac{e^{-\beta\tau} - \beta\tau_{sm}\, e^{-\tau/\tau_{sm}}}{1 - \beta\tau_{sm}} \qquad (\tau \geqslant 0).$$

Applying formula (1.9), we find that the mean number of peaks of the smoothed process is

$$n_0 = \frac{\sqrt{R_2}}{2\pi}\, e^{-b^2/2\sigma^2(y)},$$

where

$$\sigma^2(y) = \frac{\sigma^2(x)}{1 + \beta\tau_{sm}}, \quad R_2 = -R_y''(0) = \frac{\beta}{\tau_{sm}}.$$

This shows that as the smoothing time approaches zero, the number of peaks increases without limit.

Despite the fact that an exact Markov process has infinitely many points of intersection with a given level $x=b$, from time to time finite nonzero intervals of any preassigned duration do appear between these points of intersection. (This will be confirmed by the considerations of Chap. 2.) It follows that the points of intersection can be grouped into "clusters," each of which contains infinitely many such points but is of finite duration. In other words, because of its "jaggedness," the trajectory of a Markov process cannot intersect a given level just once, but rather all of a sudden produces a cluster of infinitely many intersections. If the process is smoothed, the number of intersections in each cluster becomes finite and decreases as τ_{sm} gets larger. Although the number of intersections in each cluster is strongly dependent on the smoothing time, the number of clusters and their arrangement is not, and a cluster will survive the smoothing operation if just one peak in the cluster is of duration greater than τ_{sm}. (In Chap. 2, Sec. 5, we shall examine in detail the relation between the peaks of a Markov process and those of processes obtained from it by smoothing.) Since the peaks in each cluster follow one another too closely to be distinguished by a counting device, it is often of interest to calculate the rate of arrival of "peak clusters" rather than the total number of peaks. We now devote our attention to this problem.

It is meaningful to group a sequence of peaks into clusters only if the clusters are very "clumpy," i.e., if the mean time interval T_i between clusters is much greater than the duration T_c of the clusters themselves:

$$T_i \gg T_c. \tag{1.10}$$

This will be the case if the threshold $x=b$ lies in a region seldom reached by the coordinate $x(t)$, which means that the probability $w_{st}(b)$ is relatively small.[2] Since we have the estimate

$$w_{st}(x) \sim \frac{1}{\sigma(x)} \tag{1.11}$$

in the region where $w_{st}(x)$ is large, this condition, which in a certain

[2] For the meaning of $w_{st}(x)$, see p. 1.66.

sense is equivalent to the inequality (1.10), can be written as

$$w_{st}(b) \ll \frac{1}{\sigma(x)}. \tag{1.12}$$

A more accurate condition for the applicability of the present considerations will be given later.

Now suppose we are trying to determine the mean number of peak clusters of a Markov process $x(t)$ which can be described by the fluctuation equation

$$\dot{x} = f(x) + \xi, \tag{1.13}$$

where $\xi(t)$ is a delta-correlated random process, with mean zero and correlation function

$$\langle \xi \xi_\tau \rangle = K\delta(\tau). \tag{1.14}$$

For an actual process $\xi(t)$, which is not completely delta-correlated and has a correlation time of order τ_{sm}, the intensity coefficient K is given by the integral

$$K = \int_{-\infty}^{\infty} \langle \xi \xi_\tau \rangle d\tau.$$

The function $f(x)$ in equation (1.13) can be related to a "potential function" $U(x)$ by writing

$$f(x) = -\frac{dU(x)}{dx}.$$

We assume that $U(x)$ becomes infinite as $|x| \to \infty$, which means that there is a "restoring force" preventing the coordinate $x(t)$ from penetrating the region of infinitely large positive or negative values.

Instead of (1.13), we might consider the more general fluctuation equation

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\xi, \tag{1.15}$$

describing a one-dimensional process without aftereffect. However, the change of variable

$$x = \int \frac{dx_1}{g_1(x_1)}$$

carries (1.15) into the form (1.13), so that the process $x_1(t)$ reaches the level b_1 if and only if the process $x(t)$ reaches the level

$$b = \int \frac{dx_1}{g_1(x_1)}\bigg|_{x_1=b_1}.$$

Therefore, in our calculations we can go over from (1.15) to the simpler equation (1.13).

According to the results of Chap. I.4, equation (1.13) is equivalent to the Fokker-Planck equation

$$\dot{w} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\, w\right) + \frac{K}{2}\frac{\partial^2 w}{\partial x^2}. \tag{1.16}$$

Using (1.16), we can immediately find the stationary distribution $w_{st}(x)$, satisfying the condition $\dot{w}_{st}(x)=0$. In fact, it follows from (I.4.49) that

$$\begin{aligned} w_{st}(x) &= N^{-1}\exp\left\{-\frac{2}{K}\,U(x)\right\}, \\ N &= \int \exp\left\{-\frac{2}{K}\,U(x)\right\} dx. \end{aligned} \tag{1.17}$$

To calculate the mean repetition frequency of peak clusters, we try to find the reciprocal of this quantity, i.e., the mean time interval between first crossings of the given level by neighboring clusters. Suppose that at the time $t=t_1$ (or, for simplicity, at $t=0$), we observe that the level $x=b$ is first crossed by one of the clusters. The conditional distribution of $x(t)$ at subsequent times $t>0$ spreads out from a delta function

$$w[x(t)\,|\,x(0)=b] = \delta[x(t)-b] \quad \text{for} \quad t=0 \tag{1.18}$$

to the stationary distribution

$$w[x(t)\,|\,x(0)=b] = w_{st}(x) \quad \text{for} \quad t \gg T_{tr}. \tag{1.19}$$

Here the time constant T_{tr} characterizes the duration of the transient process, i.e., the rapidity with which the stationary distribution is established. Any crossings of the threshold $x=b$ during this time will be grouped into a single cluster, and hence

$$T_c \lesssim T_{tr},$$

since, after the stationary distribution is established, the coordinate $x(t)$ will spend quite a long time in the region $x-\langle x\rangle\sim\sigma(x)$, where most of the probability (1.17) is concentrated. Only after a large mean time interval T_{fp}, called the *mean first-passage time*, will $x(t)$ again arive at the level b, thereby creating a new cluster of intersections. Therefore, our previous condition (1.10) is essentially equivalent to the condition

$$T_{fp}\gg T_{tr}. \tag{1.20}$$

These considerations imply that the mean interval between first crossings by consecutive clusters is approximately equal to the mean interval between the establishment of a stationary distribution and the time when the boundary $x=b$ is first reached. As shown in Chap. I.4, Sec. 6, such first-passage time problems are handled by solving the Fokker-Planck equation subject to the zero boundary condition

$$w(b,t)=0. \tag{1.21}$$

In so doing, we note that the normalization condition

$$\int w(x,0)\,dx=1$$

is satisfied only at the initial time $t=0$, when $w(x, t)$ coincides with the initial distribution

$$w(x,0)=w_{st}(x), \tag{1.22}$$

which is stationary in the present case. At subsequent times, the quantity

$$W(t)=\int_{-\infty}^{b} w(x,t)\,dx \tag{1.23}$$

is less than unity, and is just the probability that the curve $x(t)$ fails to reach the value $x=b$ during the time interval $[0, t]$. In other words, $W(t)$ describes that fraction of the realizations of the process $x(t)$ which have not yet managed to reach the boundary $x=b$. As time goes on, $W(t)$ decreases due to the fact that more and more realizations are excluded from consideration as a result of having reached the boundary (cf. p. I.80). The probability that first contact with the

boundary is made during the interval $[t, t+dt]$ equals

$$dP = -dW = -\dot{W}\,dt = -dt\int_{-\sigma}^{b} w(x,t)\,dx. \tag{1.24}$$

Using (1.16), we find at once that

$$dP = [G(b,t) - G(-\infty,t)]\,dt, \tag{1.25}$$

where

$$G(x,t) = -\frac{\partial U}{\partial x}w - \frac{K}{2}\frac{\partial w}{\partial x} \tag{1.26}$$

is the probability current across the abscissa x (cf. p.1.62). Dropping the current $G(-\infty, t)$ at infinity, because of the impossibility of the coordinate taking infinite values, we easily see that

$$\dot{W}(t) = -G(b,t), \tag{1.27}$$

i.e., the rate of decrease of probability equals the probability current $G(b, t)$ through the boundary $x=b$.

The probability (1.24) is the probability that first passage occurs between t and $t+dt$, and hence the quantity $-\dot{W}$ is just the probability density of the first-passage time. The mean first-passage time is found by averaging and integrating by parts:

$$T_{fp} = -\int_0^\infty t\dot{W}\,dt = \int_0^\infty W(t)\,dt. \tag{1.28}$$

According to (I.4.113) and (I.4.111), the quantity (1.28), involving the solution of the differential equation (1.16), subject to the conditions (1.21) and (1.22), can be obtained in exact form in terms of quadratures. However, here it is pointless to find a complicated exact expression for T_{fp}, since the very statement of the problem is already associated with approximations based on the condition (1.10) or (1.20). Therefore, we shall now find a simpler approximate solution of the problem, valid under these conditions.

Applying the method presented in Chap. 1.4, Sec. 4, we look for a solution of equation (1.16), subject to the boundary condition (1.21)

and initial condition (1.22), in the form of an expansion

$$w(x,t) = \sum_{m=0}^{\infty} T_m\, e^{-\lambda_m t} X_m(x). \tag{1.29}$$

Here the values of T_m are determined from (1.22), while, according to (I.4.50), the eigenfunction $X_m(x)$ satisfies the equation

$$\frac{K}{2}\frac{\partial^2 X_m}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x} X_m\right) + \lambda_m X_m = 0, \tag{1.30}$$

subject to the boundary conditions

$$X_m(b) = 0, \quad G[X_m]_{x=-\infty} = 0. \tag{1.31}$$

The boundary condition (1.21) differs from the boundary condition $w_{st}(\infty, t) = 0$ or $G[w_{st}]_{x=\infty} = 0$ satisfied by w_{st}. However, according to (1.12), this difference is not large. If b approaches infinity, where $w_{st} = 0$, the difference between the two boundary conditions disappears, and the functions satisfying (1.30) and (1.31) go into the corresponding functions X_m, defined for all values of x, which were considered in Chap I.4, Sec. 4. In particular, according to (I.4.53), X_0 goes into w_{st}, and λ_0 vanishes. It follows that for sufficiently large b, satisfying the condition (1.12), X_0 differs only slightly from w_{st} in the region of large probability, and λ_0 differs only slightly from 0 ($\lambda_0 \ll \lambda_1, \lambda_2, \ldots$). After a time t much greater than the time T_{tr} required for the transients to die out and the stationary distribution to become established, i.e., for $t \gg T_{tr} \sim 1/\lambda_1$, the only important term in the expansion (1.29) is

$$w(x,t) = T_0 X_0(x)\, e^{-\lambda_0 t}. \tag{1.32}$$

Moreover, because of the initial condition (1.22) and the closeness of $X_0(x)$ to $w_{st}(x)$ noted above, we have $T_0 \approx 1$.

Thus, for $t \gg T_{tr}$, a "quasi-static" regime is established, characterized by an exponential decay of the probability density. However, the shape of the probability density remains unchanged, and is described by the function $X_0(x)$ satisfying the equation

$$\frac{K}{2}\frac{\partial^2 X_0}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x} X_0\right) = -\lambda_0 X_0. \tag{1.33}$$

The total probability (1.23) equals

$$W(t) = e^{-\lambda_0 t}, \tag{1.34}$$

since

$$\int_{-\infty}^{b} X_0\, dx \approx \int_{-\infty}^{b} w_{st}\, dx \approx 1, \tag{1.35}$$

and hence falls off slowly to zero. Formula (1.28) then gives

$$T_{fp} = \frac{1}{\lambda_0}. \tag{1.36}$$

Integrating (1.33) from $-\infty$ to x, we obtain

$$\frac{K}{2}\frac{\partial X_0}{\partial x} + \frac{\partial U}{\partial x} X_0 = -\lambda_0 \int_{-\infty}^{x} X_0(x')\,dx'. \tag{1.37}$$

A second integration gives

$$X_0(x) = \frac{2\lambda_0}{K} \int_{x}^{b} \exp\left\{-\frac{2}{K}\left[U(x) - U(x')\right]\right\} g(x')\,dx', \tag{1.38}$$

where

$$g(x) = \int_{-\infty}^{x} X_0(x')\,dx' \approx \int_{-\infty}^{x} w_{st}(x')\,dx'. \tag{1.39}$$

Using the stationary distribution (1.17), we can write (1.38) in the form

$$X_0(x) = \frac{2\lambda_0}{K}\, w_{st}(x) \int_{x}^{b} \frac{g(x')}{w_{st}(x')}\, dx'. \tag{1.40}$$

It follows from the inequality (1.12) that the integrand g/w_{st} is large in the region near b and small in the region of large probability $x = x_1$, where

$$w_{st}(x_1) \sim \frac{1}{\sigma(x)}.$$

In addition, we assume that

$$\int_{x_1}^{b} \frac{dx'}{w_{st}} \gg \int_{x_1-\sigma(x)}^{x_1} \frac{dx'}{w_{st}} \sim \sigma^2(x). \tag{1.41}$$

Then the integral

$$\int_{x_1}^{b} \frac{g}{w_{st}}\, dx'$$

is weakly dependent on x_1, provided that x_1 lies in the region of large probability. Moreover, the expression g/w_{st} is close to $1/w_{st}$ in the region where it is large, since there

$$g(x) \approx \int_{-\infty}^{b} w_{st}\, dx' \approx 1, \tag{1.42}$$

as can be seen from (1.39).

Thus, choosing $x = x_1$ in the region of large probability, and bearing in mind that in this region $X_0 \approx w_{st}$, we find from (1.40), after replacing g by 1, that

$$\frac{1}{\lambda_0} = \frac{2}{K} \int_{x_1}^{b} \frac{dx'}{w_{st}(x')}, \tag{1.43}$$

or, more explicitly,

$$\frac{1}{\lambda_0} = 2\,\frac{N}{K} \int_{x_1}^{b} \exp\left\{\frac{2}{K}\, U(x)\right\} dx. \tag{1.44}$$

This is the desired expression for the mean repetition rate of peak clusters. Because of (1.41), the expression (1.44) is practically independent of x_1. According to (1.43), the basic condition (1.41) for applicability of our theory can be written in the form

$$\frac{1}{\lambda_0} \gg \frac{2\sigma^2(x)}{K}. \tag{1.45}$$

In view of (1.36), it is easy to see that this inequality is equivalent to

the condition (1.20), since

$$T_{\eta} \sim \frac{K}{2\sigma^2(x)}$$

(cf. pp.I.22, I.132).

3. Application of the Formula for the Mean Repetition Rate of Peak Clusters

We now give some examples illustrating the application of formula (1.44).

Example 1. Consider the nonlinear restoring force

$$f(x) = -f_0 \frac{x}{|x|}, \tag{1.46}$$

corresponding to the potential function

$$U(x) = f_0 |x| . \tag{1.47}$$

According to (1.17), in this case we have

$$w_{st}(x) = \frac{1}{N} \exp\left\{-\frac{2}{K} f_0 |x|\right\}, \quad N = \frac{K}{f_0}, \quad \langle x \rangle = 0, \quad \sigma(x) = \frac{K}{2f_0}. \tag{1.48}$$

Applying (1.44), we find that

$$\frac{1}{\lambda_0} = \frac{N}{f_0} (e^{2f_0 b/K} - e^{2f_0 x_1/K}). \tag{1.49}$$

The condition for the validity of the last result, which can be written in any of the three forms (1.10), (1.20) or (1.45), then becomes

$$K \ll 2f_0 b,$$

and actually implies that (1.49) is weakly dependent on x_1 if $b - x_1 \gg \sigma(x)$. Neglecting the term $e^{2f_0 x_1/K}$, we obtain the following expression for the mean repetition rate of the peak clusters:

$$\lambda_0 = \frac{f_0^2}{K} \exp\left\{-\frac{2}{K} f_0 b\right\}. \tag{1.50}$$

Example 2. Next consider the case of a linear restoring force:

$$f(x) = -\beta x, \quad U(x) = \frac{\beta}{2} x^2. \tag{1.51}$$

In this case, the stationary distribution is Gaussian, i.e.,

$$w_{st}(x) = \frac{1}{N} e^{-\beta x^2/K}, \quad N = \sqrt{\frac{\pi K}{\beta}}, \quad \sigma(x) = \sqrt{\frac{K}{2\beta}}, \tag{1.52}$$

and (1.44) gives

$$\frac{1}{\lambda_0} = 2\sqrt{\frac{\pi}{\beta K}} \int_0^b e^{\beta x^2/K}\, dx, \tag{1.53}$$

where for x_1 we choose the mean value $\langle x \rangle = 0$. The condition for applicability of our theory is now the inequality $\beta b^2/K \gg 1$, which allows us to use the asymptotic series

$$\int_0^b e^{\beta x^2/K}\, dx = \frac{K}{2\beta b} e^{\beta b^2/K} \left(1 + \frac{K}{2\beta b^2} + \dots\right). \tag{1.54}$$

Retaining only the first term of this expansion, we find the following expression for the mean repetition rate of the peak clusters:

$$\lambda_0 = \beta b \sqrt{\frac{\beta}{\pi K}}\, e^{-\beta b^2/K}. \tag{1.55}$$

It should be noted that the key factors in formulas (1.50) and (1.55), i.e., the exponentials

$$\exp\left\{-\frac{2}{K} f_0 b\right\} \quad \text{and} \quad \exp\left\{-\frac{\beta b^2}{K}\right\},$$

are nothing other than

$$\exp\left\{-\frac{2}{K} U(b)\right\}.$$

Such a factor is always obtained in cases where the expression

$$\exp\left\{\frac{2}{K} U(x)\right\}$$

can be replaced by the first two terms of its Taylor series:

$$\exp\left\{\frac{2}{K}U(b)-\frac{2}{K}f(b)(x-b)\right\}.$$

After making this substitution and replacing the lower limit of integration by $-\infty$, we find that

$$\lambda_0=-\frac{f(b)}{N}\exp\left\{-\frac{2}{K}U(b)\right\}. \tag{1.56}$$

The fact that (1.50) and (1.55) are special cases of (1.56) is apparent at once. However, (1.56) does not cover many cases where the derivative

$$\frac{dU(b)}{dx}=-f(b)$$

is insufficiently large, or cases where the derivative is even negative. Then an important role is played by the nonlinear term

$$\frac{1}{2}\frac{d^2U(b)}{dx^2}(x-b)^2.$$

(One such case will be considered in Chap. 9, Sec. 3). Cases can also arise where a Taylor series expansion of the function $U(x)$ is altogether inapplicable.

Example 3. Suppose the "force" and "potential" have the form

$$f(x)=-\beta x+\frac{K}{2x},\quad U(x)=\frac{\beta}{2}x^2-\frac{K}{2}\ln x. \tag{1.57}$$

Then $x(t)$ is a Rayleigh process (see pp. I.73, I.192–I.195), with the stationary distribution

$$w_{st}(x)=\frac{x}{\sigma^2}e^{-x^2/2\sigma^2}\qquad\left(\sigma^2=\frac{K}{2\beta}\right). \tag{1.58}$$

Unlike the preceding examples, $x(t)$ can now take positive values only, and moreover small values $x\ll\sigma$ are just as unlikely as large values $x\gg\sigma$. Thus, the condition for applicability of our theory will be satisfied if we choose a very low level $b\ll\sigma$. This allows us to calculate the

number of clusters of intersection points, provided suitable slight modifications are made in the formulas derived above, corresponding to the fact that the region of probable values of the coordinate lies *above* (rather than below) the threshold level. Using formula (1.43) and choosing $x_1 = \sigma$, we find that

$$\frac{1}{\lambda_0} = \frac{1}{\beta} \int_b^\sigma e^{x^2/2\sigma^2} \frac{dx}{x}. \tag{1.59}$$

Here the main contribution to the integral is made by the region near b, where $e^{x^2/2\sigma^2} \approx 1$. Replacing the exponential by unity, we obtain

$$\lambda_0 = \frac{\beta}{\ln(\sigma/\beta)}, \tag{1.60}$$

and the condition (1.45) for applicability of the theory becomes

$$\ln \frac{\sigma}{\beta} \gg 1. \tag{1.61}$$

In conclusion, we say a few words about the condition (1.10) underlying the above considerations. The condition is satisfied when the threshold b is relatively high, or when the "scatter" of the noise fluctuations is small. This is the case of greatest practical interest, since the noise level in a device intended for a given application, such as counting pulses, must not be so high as to prevent the basic operation of the device. In fact, the perturbation due to the noise must be relatively small, for otherwise the excessive number of false peaks would render normal operation of the device impossible and make it unsuitable for practical purposes. Therefore, we can assume that the "small-fluctuation condition" (1.10) is satisfied in most engineering situations.

CHAPTER 2

The Duration of Peaks of a Markov Process

In many problems, it is not enough to know the mean number of peaks, since the system under consideration can often respond only to peaks satisfying certain conditions. One such condition involves the duration of the peaks. Suppose there is a certain minimum peak duration τ such that peaks of smaller duration "go unnoticed." Then, obviously, the quantity of interest is not the average total number of peaks n_0 (per unit time) but rather the average number of peaks $n(\tau)$ whose duration exceeds τ.

Suppose the total number of peaks is finite and known, and suppose we have managed to find $w(\tau)$, the probability density of the distribution of peaks with respect to duration. Then $n(\tau)$ is given by the formula

$$n(\tau) = n_0 \int_{\tau}^{\infty} w(\tau)\, d\tau \qquad [n_0 = n(0)]. \tag{2.1}$$

In the case of Markov processes, the total number of peaks is infinite, because of the arbitrarily large number of arbitrarily short peaks, and hence a normalized probability density $w(\tau)$ does not exist. Nevertheless, $n(\tau)$ is a well-defined and finite quantity, which we shall calculate in this chapter.

As will be shown in Sec. 5, an actual smoothly varying random process, which can be obtained by subjecting a neighboring Markov process to smoothing with a small time constant τ_{sm}, has almost the

same number of peaks of duration $\tau \gg \tau_{sm}$ as the Markov process itself. Therefore, the results to be presented here apply not only to Markov processes but also to the "nearly-Markovian" smoothly varying processes encountered in practice.

1. The Mean Number of Peaks of Duration Exceeding τ

We again consider a Markov process satisfying the fluctuation equation[1]

$$\dot{x} = f(x) + \xi, \tag{2.2}$$

where $\xi(t)$ is a delta-correlated random process, with mean zero and correlation function

$$\langle \xi \xi_\tau \rangle = K\delta(\tau). \tag{2.3}$$

Corresponding to (2.2), we have the Fokker-Planck equation

$$\dot{w} = -\frac{\partial}{\partial x}[f(x)w] + \frac{K}{2}\frac{\partial^2 w}{\partial x^2}, \tag{2.4}$$

which describes the way the probability density $w(x, t)$ evolves in time.

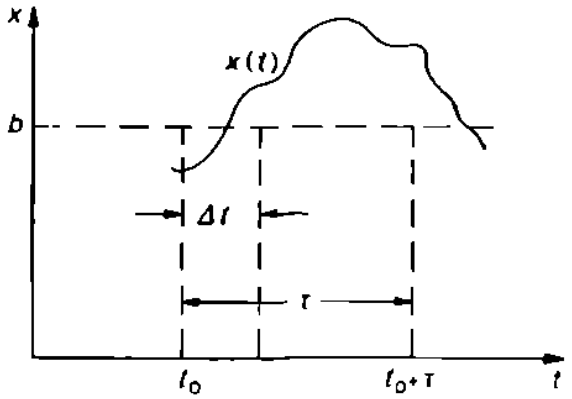

Fig. 2. Beginning and end of a peak.

[1] Recall that a change of variable converts the fluctuation equation (1.15) to this form, so that (2.3) is actually quite general.

Our problem is to calculate the quantity $n(\tau)$, the mean density of peaks whose durations exceed a given positive number τ. Consider a small interval $[t_0, t_0+\Delta t]$, where $\Delta t<\tau$, and let Δn be the mean number of peaks which begin in this interval and end at a time $t>t_0+\tau$ (see Figure 2). For simplicity, we henceforth set $t_0=0$. Once we have calculated Δn, the required mean density of peaks can be found by taking the limit

$$n(\tau)=\lim_{\Delta t\to 0}\frac{\Delta n}{\Delta t}. \tag{2.5}$$

For every realization of the process, there can be no more than one peak of the type described, and in fact, Δn is just the probability of occurrence ΔP of such a peak. It should be noted that if a peak begins in the interval $[0, \Delta t]$ and ends at a time $t>\tau$, then the trajectory crosses the level $x=b$ at least once during the interval $[0, \Delta t]$, and exceeds this level during the interval $[\Delta t, \tau]$, i.e.,

$$x(t)>b \quad \text{for} \quad \Delta t\leqslant t\leqslant\tau. \tag{2.6}$$

We now divide the trajectories of the process $x(t)$ into two classes, the first consisting of all trajectories which cross the level b during the interval $[0, \Delta t]$, and the second consisting of all other trajectories. Suppose the trajectories of both kinds are described by a probability density $w_0(x)=w(x, 0)$, satisfying equation (2.4.) In particular, if the process $x(t)$ is stationary, then $w_0(x)$ is the stationary distribution (1.17). The probability density $w_1(x, t)$, describing the trajectories of the first class, which never cross the level b, must satisfy the same equation (2.4) with the following obvious initial and boundary conditions:

$$\begin{aligned} &w_1(x,0)=w(x,0) && (x>b),\\ &w_1(b,t)=0 && (0\leqslant t\leqslant\Delta t). \end{aligned} \tag{2.7}$$

The trajectories of the second class, which are left after eliminating trajectories of the first class, are described by the probability density

$$w_2(x,t)=w_0(x)-w_1(x,t), \tag{2.8}$$

which also satisfies equation (2.4):

$$\dot{w}_2 = -\frac{\partial}{\partial x}[f(x)w_2] + \frac{K}{2}\frac{\partial^2 w_2}{\partial x^2}. \tag{2.9}$$

Using (2.7), we find that (2.8) satisfies the initial condition

$$w_2(x,0) = 0 \quad (x > b) \tag{2.10}$$

and the boundary condition

$$w_2(b,t) = w_0(b) \quad (0 \leqslant t \leqslant \Delta t). \tag{2.11}$$

The trajectories of the second class which lie above the level $x=b$ at time $t=\Delta t$ have clearly crossed this level at least once. Therefore the fraction of trajectories with a peak beginning in the interval $[0, \Delta t]$ which does not end by the time $t=\Delta t$ is equal to

$$\int_b^\infty w_2(x, \Delta t)\,dx. \tag{2.12}$$

This is also the probability of occurrence of such a peak. To find the probability that a peak once having begun does not end by time $t=\tau$, we have to take account of the loss of trajectories which cross the level $x=b$ at least once during the interval $[\Delta t, \tau]$. As usual, this is done by solving equation (2.4) subject to the zero boundary condition

$$w_3(b,t) = 0 \quad (t > \Delta t) \tag{2.13}$$

and the initial condition

$$w_3(b, \Delta t) = w_2(b, \Delta t). \tag{2.14}$$

Having solved this problem, we can find the probability ΔP that a peak begins in the interval $[0, \Delta t]$ and does not end by the time $t=\tau$:

$$\Delta P = \Delta n = \int_b^\infty w_3(x, \tau)\,dx. \tag{2.15}$$

Since the density w_2 which is the solution of the problem (2.9)–(2.11) goes over continuously into the density w_3 whose behavior for $t > \Delta t$

is described by (2.4), (2.13) and (2.14), it is convenient to combine them by writing

$$w_3(x,t) \equiv w_2(x,t) \quad (t > \Delta t). \tag{2.16}$$

After making this identification, the function $w_2(x, t)$ satisfies equation (2.4), with the initial condition

$$w_2(x,0) = 0 \quad (x > b) \tag{2.17}$$

and boundary condition

$$w_2(b,t) = \begin{cases} w_0(b) & \text{for} \quad 0 \leqslant t \leqslant \Delta t, \\ 0 & \text{for} \quad \Delta t < t. \end{cases} \tag{2.18}$$

According to (2.5) and (2.15), the limit of the ratio $w_2/\Delta t$ as $\Delta t \to 0$ will be needed later. This suggests introducing the function $v(x, t)$, defined by the formula

$$w_0(b)\, v(x,t) = \lim_{\Delta t \to 0} \frac{w_2(x,t)}{\Delta t} \quad (x > b). \tag{2.19}$$

Using (2.9), (2.17) and (2.18), we find that $v(x, t)$ satisfies the equation

$$\dot{v} = -\frac{\partial}{\partial x}\left[f(x)\, v\right] + \frac{K}{2}\frac{\partial^2 v}{\partial x^2}, \tag{2.20}$$

subject to the conditions

$$v(x,t) = 0 \quad (t < 0), \tag{2.21}$$

$$v(b,t) = \delta(t). \tag{2.22}$$

After solving (2.20) with the conditions (2.21) and (2.22), the required quantity $n(\tau)$ is found from the formula

$$n(\tau) = w_0(b) \int_b^\infty v(x,\tau)\, dx, \tag{2.23}$$

implied by (2.5), (2.15) and (2.19). The problem just posed is conveniently solved by using the Carson-Laplace transform.[2] In fact,

[2] See H. S. Carslaw and J. C. Jaeger, *Operational Methods in Applied Mathematics*, second edition, Oxford University Press, London (1953), p. xv.

the usual rules lead to the formula

$$\frac{\partial^2 \bar{v}}{\partial x^2} - \frac{2}{K}\frac{\partial}{\partial x}(f\bar{v}) - \frac{2p}{K}\bar{v} = 0, \tag{2.24}$$

where the overbar denotes the transform

$$\bar{v}(x,p) = p\int_{0-}^{\infty} e^{-pt}v(x,t)\,dt. \tag{2.25}$$

Then the boundary condition (2.22) takes the form

$$\bar{v}(b,p) = p. \tag{2.26}$$

The solution of (2.24) contains two arbitrary constants of integration. One of these is determined from (2.26), and the other can be determined from the condition that the function $\bar{v}(x,p)$ vanish at infinity:

$$\bar{v}(x,p) \to 0 \quad \text{as} \quad x \to \infty \qquad (\text{Re } p > 0). \tag{2.27}$$

Once we have solved (2.24) for $\bar{v}$, we can easily calculate $n(\tau)$. In fact, according to (2.23), the transform of $n(\tau)$, i.e., the function

$$\bar{n}(p) = p\int_{0}^{\infty} e^{-pt}n(t)\,dt, \tag{2.28}$$

is given by

$$\bar{n}(p) = w_0(b)\int_{b}^{\infty} \bar{v}(x,p)\,dx. \tag{2.29}$$

This formula can be written in another form. Integrating the expression

$$\bar{v} = \frac{1}{p}\frac{\partial}{\partial x}\left(\frac{K}{2}\frac{\partial \bar{v}}{\partial x} - f\bar{v}\right)$$

implied by (2.24), and using (2.26) and the fact that the current

$$\bar{v} - \frac{K}{2}\frac{\partial \bar{v}}{\partial x}$$

vanishes at infinity, we find that

$$\tilde{n}(p) = -\frac{K}{2p} w_0(b) \frac{\partial \bar{v}(b,p)}{\partial x} + f(b) w_0(b). \tag{2.30}$$

The second term on the right plays no special role, since it merely corresponds to the constant $f(b)w_0(b)$ after transforming back to $n(\tau)$. The function $n(\tau)$ corresponding to (2.30) can conveniently be found by using tables of integral transforms.[3]

The above method is also suitable for calculating the mean number of intervals between peaks which exceed a given positive number τ. We denote this quantity by $n_2(\tau)$, as opposed to $n_1(\tau) = n(\tau)$. The only difference between the two quantities is that we are now interested in the case $x(t) < b$ rather than $x(t) > b$. Each case goes into the other under the substitution $x \to -x$.

2. Examples

We now give some examples illustrating the theory developed in the preceding section.

Example 1. The simplest situation corresponds to the absence of any restoring force at all:

$$f(x) = 0. \tag{2.31}$$

In this case, the process $x(t)$ is nonstationary, but the theory remains the same as before, except that the function $w_0(x)$ now depends on the time t_0. Equation (2.24) takes the form

$$\frac{\partial^2 \bar{v}}{\partial x^2} - \frac{2p}{K} \bar{v} = 0, \tag{2.32}$$

[3] See Bateman Manuscript Project, *Tables of Integral Transforms, Vol. I*, McGraw-Hill Book Co., Inc., New York (1954), esp. Chaps. 4 and 5. Also see I. M. Ryshik and I. S. Gradsteln, *Tables of Series, Products, and Integrals*, VEB Deutscher Verlag der Wissenschaften, Berlin (1957), p. 254 ff.

and has the general solution

$$C_1 \exp\left\{\frac{2p}{K}\, x\right\} + C_2 \exp\left\{-\frac{2p}{K}\, x\right\}. \tag{2.33}$$

Choosing constants C_1 and C_2 which satisfy the conditions (2.26) and (2.27), we find that

$$\bar{v}(x, p) = p \exp\left\{\frac{2p}{K}\,(b - x)\right\}.$$

Substitution of this expression into (2.30) gives

$$\bar{n}(p) = \sqrt{\frac{pK}{2}}\, w_0(b), \tag{2.34}$$

and hence, according to (2.28),[4]

$$n(\tau) = w_0(b) \sqrt{\frac{K}{2\pi\tau}}. \tag{2.35}$$

Thus the number of peaks of duration greater than a fixed positive number τ is indeed finite, but at the same time, the total number of peaks, obtained by taking the limit $\tau \to 0$, is infinitely large.

Example 2. Next let the absolute value of the coordinate be bounded by the number $c > b$, and suppose no restoring force acts in the interval $-c < x < c$. Then the point $x(t)$ moves as if it were free in the "bottom of a potential well." The corresponding potential function

$$U(x) = -\int^{x} f(x')\, dx'$$

has the form shown in Figure 3, and can be represented analytically by

$$U(x) = \begin{cases} 0 & \text{for } |x| < c, \\ \infty & \text{for } |x| > c. \end{cases}$$

The stationary distribution is found to be

$$w_0(x) = \begin{cases} \dfrac{1}{2c} & \text{for } |x| < c, \\ 0 & \text{for } |x| > c. \end{cases}$$

[4] See Bateman Manuscript Project, *op. cit.*, formula (21), p. 235.

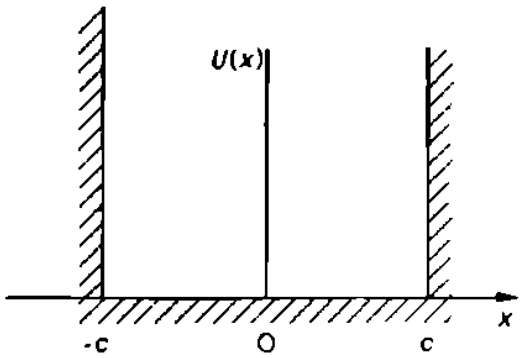

Fig. 3. The potential function for Example 2.

Equation (2.32) continues to hold inside the well, because of the absence of a restoring force, but the values of the constants in the general solution are now different. In fact, the probability current vanishes at the reflecting wall $x = c$, and hence

$$\frac{\partial \bar{v}(c, p)}{\partial x} = 0. \tag{2.36}$$

It follows from (2.26) and (2.36) that

$$\bar{v}(x, p) = p \frac{\cosh \sqrt{\frac{2p}{K}}\,(c - x)}{\cosh \sqrt{\frac{2p}{K}}\,(c - b)}$$

Formula (2.29) now gives

$$\bar{n}(p) = \sqrt{\frac{pK}{2}}\, w_0(b) \tanh \sqrt{\frac{2p}{K}}\,(c - b). \tag{2.37}$$

Therefore[5]

$$n(\tau) = \frac{K w_0(b)}{2(c - b)}\, \vartheta_2\left(0, \frac{K\tau}{2(c - b)^2}\right), \tag{2.38}$$

where

$$\vartheta_2(0, x) = 2 \sum_{k=0}^{\infty} e^{-\pi^2 (k + \frac{1}{2})^2 x}.$$

[5] See I. M. Ryshik and I. S. Gradstein, *op. cit.*, formula (5.241.2), p. 261, taking account of formula (6.180.3), p. 287 and Sec. 6.186, p. 288.

Example 3. We now consider the case of a constant restoring force[6]

$$f(x) = -f_0 \quad (x > b, \quad f_0 > 0). \tag{2.39}$$

Then equation (2.24) becomes

$$\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{2f_0}{K}\frac{\partial \bar{v}}{\partial x} - \frac{2p}{K}\bar{v} = 0. \tag{2.40}$$

Of the two linearly independent solutions

$$\exp\left\{-(f_0 + \sqrt{f_0^2 + 2pK})\frac{x}{K}\right\}, \quad \exp\left\{-(f_0 - \sqrt{f_0^2 + 2pK})\frac{x}{K}\right\}$$

of this equation, only the first satisfies the condition (2.27). Taking account of (2.26), we obtain

$$\bar{v}(x, p) = p \exp\left\{\frac{b - x}{K}(f_0 + \sqrt{f_0^2 + 2pK})\right\},$$

which gives

$$\bar{n}(p) = w_0(b)\frac{pK}{f_0 + \sqrt{f_0^2 + 2pK}}, \tag{2.41}$$

after substitution into (2.29). Using a table of integral transforms, we find that[7]

$$n(\tau) = \frac{w_0(b)}{2}\left\{\frac{e^{-f_0^2\tau/2K}}{\sqrt{\pi\tau/2K}} - f_0 \operatorname{erfc}\left(f_0\sqrt{\frac{\tau}{2K}}\right)\right\}, \tag{2.42}$$

where

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}\,dy. \tag{2.43}$$

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[6] Cf. (1.46) and the stationary distribution (1.48).

[7] See Bateman Manuscript Project, *op. cit.*, formula (4), p. 233, noting that

$$\frac{\bar{n}(p)}{p} = \sqrt{\frac{K}{2}}\frac{w_0(b)}{\beta}\frac{1}{1 + \sqrt{p'}},$$

where

$$p' = 1 + \frac{p}{\beta^2}, \quad \beta = \frac{f_0}{\sqrt{2K}}.$$

These results can also be used when $f_0<0$, i.e., when the force causes the trajectory to move away from the threshold level. In this case, the function (2.42) does not approach zero as $\tau\to\infty$. This means that some of the peaks which begin in the interval $[0, \Delta t]$ never end, because the curve goes off to infinity, once having exceeded the threshold. According to (2.42), the mean number of peaks of this kind is

$$n(\infty)\Delta\tau = w_0(b)|f_0|\Delta\tau. \tag{2.44}$$

Example 4. Of particular importance is the case of a linear restoring force

$$f(x) = -\beta x = -\frac{x}{a},$$

which, as we know (see p.I.106) leads to a Gaussian Markov process with correlation function

$$\langle xx_t\rangle = \frac{K}{2\beta}e^{-\beta|t|}$$

and stationary distribution (1.52). Equation (2.24) now takes the form

$$\frac{\partial^2\bar v}{\partial x^2} + \frac{2x}{aK}\frac{\partial\bar v}{\partial x} + \frac{2}{aK}(1-ap)\bar v = 0,$$

or

$$\frac{\partial^2\bar v}{\partial y^2} + y\frac{\partial\bar v}{\partial y} + (1-ap)\bar v = 0, \tag{2.45}$$

where we have the substitution

$$y = \sqrt{\frac{2}{aK}}\,x.$$

Comparing (2.45) with formula (7.355.3), p. 405, from Ryshik and Gradstein's handbook, we find that the linearly independent solutions of (7.45) are

$$e^{-y^2/4}D_{-ap}(y) \quad\text{and}\quad e^{-y^2/4}D_{-ap}(-y),$$

where $D_{-ap}(z)$ is the parabolic cylinder function. Only the first of these solutions vanishes as $y\to\infty$, and hence the desired solution of (2.45),

satisfying the conditions (2.26) and (2.27), can be written in the form

$$\bar{v}(x,p) = p \exp\left\{\frac{b^2-x^2}{2aK}\right\} \frac{D_{-ap}\left(\sqrt{\frac{2}{aK}}\, x\right)}{D_{-ap}\left(\sqrt{\frac{2}{aK}}\, b\right)}. \tag{2.46}$$

Substituting (2.46) into (2.30), and using formula (7.347.2), p. 404, from the reference just cited, we find that

$$\bar{n}(p) = w_0(b)\sqrt{\frac{aK}{2}}\, p \frac{D_{-ap-1}\left(\sqrt{\frac{2}{aK}}\, b\right)}{D_{-ap}\left(\sqrt{\frac{2}{aK}}\, b\right)}. \tag{2.47}$$

Before considering the case of an arbitrary threshold level, we examine two special cases:

1. If $b=0$ we can express $n(\tau)$ in terms of elementary functions. In fact, according to formula (7.341.2), p. 402, again from Ryshik and Gradstein's handbook, we have

$$D_{-q}(0) = \frac{1}{\Gamma(q)}\int_0^\infty e^{-x^2/2}x^{q-1}dx = 2^{(q/2)-1}\frac{\Gamma(q/2)}{\Gamma(q)} \quad (\operatorname{Re} q > 0). \tag{2.48}$$

Setting first $q=ap$ and then $q=ap+1$ in this formula, and substituting the results into (2.47), we find that

$$\bar{n}(p) = w_0(0)\sqrt{aK}\, p \frac{\Gamma\left(\frac{ap}{2}+\frac{1}{2}\right)\Gamma(ap)}{\Gamma(ap+1)\Gamma\left(\frac{ap}{2}\right)} = \frac{w_0(0)}{2}\sqrt{aK}\, p \frac{\Gamma\left(\frac{ap}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{ap}{2}+1\right)}, \tag{2.49}$$

where we have used the duplication formula for the gamma function. If we go over to the beta function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

using the fact that $\Gamma(\frac{1}{2})=\sqrt{\pi}$, then (2.49) becomes

$$\bar{n}(p)=\frac{w_0(0)}{2}\sqrt{\frac{\overline{aK}}{\pi}}\,pB\left(\frac{ap}{2}+\frac{1}{2},\frac{1}{2}\right). \tag{2.50}$$

Noting that[8]

$$pB(p+\tfrac{1}{2},\tfrac{1}{2})=p\int_0^\infty e^{-pt}(e^t-1)^{-1/2}\,dt, \tag{2.51}$$

we see that the inverse transform of (2.50) is just

$$n(\tau)=w_0(0)\sqrt{\frac{\overline{K}}{\pi a}}\,(e^{2\tau/a}-1)^{-1/2}. \tag{2.52}$$

or

$$n(\tau)=\frac{1}{\pi a}\,(e^{2\tau/a}-1)^{-1/2}\qquad (b=0), \tag{2.53}$$

where we have substituted from (1.52).

2. The opposite case, where the threshold is high, i.e.,

$$\frac{b^2}{aK}\gg 1\quad\text{or}\quad\frac{b^2}{\sigma^2}\gg 1 \tag{2.54}$$

is of great practical interest. Then a large restoring force

$$f(x)=-\frac{x}{a}\approx-\frac{b}{a}$$

acts on the coordinate when it exceeds the level $x=b$. As a result, the peaks are of short duration, and they exceed $x=b$ by only a small amount:

$$x(t)-b\sim\frac{aK}{b}.$$

For such excursions above the threshold, the restoring force $-x/a$ differs only slightly from $-b/a$, since, according to (2.54),

$$\left|\frac{f(x)-f(b)}{f(b)}\right|=\frac{x-b}{b}\sim\frac{aK}{b^2}\ll 1.$$

[8] Under the substitution $u=e^{-t}$, (2.51) becomes

$$pB(p+\tfrac{1}{2},\tfrac{1}{2})=p\int_0^1 u^{p-1}(1-u)^{-1/2}\,du.$$

Therefore the restoring force $f(x)$ can be regarded as practically constant, and in fact, setting

$$f_0 = \frac{b}{a}, \tag{2.55}$$

we can use the formulas derived in Example 3. Thus, with the stationary distribution (1.52), formula (2.42) gives the following mean density of peaks of duration greater than τ

$$\begin{aligned} n(\tau) &= \frac{b}{2a} w_0(b) \left[\frac{e^{-T}}{\sqrt{\pi T}} - \operatorname{erfc} \sqrt{T} \right] \\ &= \frac{b}{2a\sqrt{\pi a K}} e^{-b^2/aK} \left[\frac{e^{-T}}{\sqrt{\pi T}} - \operatorname{erfc} \sqrt{T} \right], \end{aligned} \tag{2.56}$$

where

$$T = \frac{b^2 \tau}{2a^2 K} = \frac{1}{4} \frac{b^2}{\sigma^2} \frac{\tau}{a}.$$

Naturally, the same result can be deduced from formula (2.47), but with considerably more effort.

Similarly, we can calculate the mean number $n_2(\tau)$ of intervals between peaks which exceed τ. The force $f = -\beta x$ can be regarded as constant for time intervals during which $x(t)$ manages to change only slightly compared to $\sigma(x)$, i.e., intervals τ such that

$$|f\tau| \ll \sigma \quad \text{or} \quad \tau \ll \frac{a\sigma}{b}.$$

Using (2.42) with the negative value $f_0 = -b/a$, we obtain the formula

$$n_2(\tau) = \frac{b}{2a} w_0(b) \left[\frac{e^{-T}}{\sqrt{\pi T}} + \operatorname{erfc}(-\sqrt{T}) \right], \tag{2.57}$$

where

$$T = \frac{1}{4} \frac{b^2}{\sigma^2} \frac{\tau}{a}$$

[cf. (2.56)], which is suitable for $T \ll b/\sigma$, since then $\tau \ll a\sigma/b$. It follows that the frequency of long intervals τ such that

$$\frac{a\sigma}{b} \gg \tau \gg \frac{a\sigma^2}{b^2} \qquad (T \gg 1)$$

equals

$$\frac{b}{a} w_0(b),$$

which coincides with the repetition rate (1.56) of peak clusters. Because of (1.34), the long intervals have an exponential distribution, and hence

$$n_2(\tau) \approx \frac{b}{a} w_0(b) \exp\left\{-\frac{b}{a} w_0(b)\tau\right\} \quad \left(\tau \gg \frac{a\sigma^2}{b^2}\right). \tag{2.58}$$

Asymptotic formulas like (2.56)–(2.58) can be found in all cases where the theory of peak clusters developed in Chap. 1, Sec. 2, is applicable, i.e., whenever the condition (1.45) holds.

★ Suppose now that no special assumptions are made concerning the size of the ratio b/σ. Then we must find the inverse Carson-Laplace transform of the expression (2.47). Using the fact that (2.47) is a meromorphic function, with the partial fraction expansion

$$p \frac{D_{-ap-1}}{D_{-ap}} = \sum_{m=0}^{\infty} \frac{D_{-ap_m-1}}{\left[\frac{\partial}{\partial p} D_{-ap}\right]_{p=p_m}} \frac{p}{p-p_m},$$

we have

$$n(\tau) = w_0(b) \sqrt{\frac{aK}{2}} \sum_{m=0}^{\infty} \frac{D_{a\lambda_m-1}}{-aD'_{a\lambda_m}} e^{-\lambda_m \tau} \quad \left(D'_q = \frac{\partial D_q}{\partial q}\right), \tag{2.59}$$

where the numbers $p_m = -\lambda_m$ are the roots of the equation

$$D_{-ap} = D_{a\lambda} = 0. \tag{2.60}$$

In all these formulas, the argument of the parabolic cylinder functions is taken to be

$$\sqrt{\frac{2}{aK}}\, b = \frac{b}{\sigma}.$$

The series (2.59) can also be obtained by finding the solution $w_2(x,t)$ or $v(x,t)$ of the Fokker-Planck equation (2.9) or (2.20), with zero boundary conditions, in the form of an expansion (1.29), and then integrating with respect to x from b to ∞. The roots λ_m turn out to be

the eigenfunctions of equation (1.30), while the functions $D_{a\lambda_m}$ are proportional to $X_m(x)$.

A solution of this kind, involving eigenfunctions and eigenvalues of equation (1.30), can be found for other forms of the function $f(x)$ as well. For example, in the case of Rayleigh fluctuations, the functions X_m can be expressed in terms of confluent hypergeometric functions.★

3. The Unnormalized Probability Density of the Peak Durations

Analyzing the formulas determining the number of peaks of duration exceeding a given positive number τ, we easily see that the total number of peaks, obtained by setting $\tau=0$, is infinite. Moreover, the formula

$$n(\tau) = w_0(b)\sqrt{\frac{\bar{K}}{2\pi\tau}}, \tag{2.61}$$

which is exact for $f(x)\equiv 0$, holds for sufficiently small τ, regardless of the concrete form of the process. In fact, if $\sqrt{\bar{K}}\tau \gg f(b)\tau$, then the contribution of $f(b)$, the regular component of the velocity of displacement, does not have much of an effect. Therefore, since the majority of peaks of duration exceeding τ are short (of order τ), (2.61) is a good approximation if

$$\tau \lesssim \frac{K}{f^2(b)}.$$

The fact that $n(0)$ is infinite prevents us from using formula (2.61), defining the normalized probability density $w(\tau)$ of the peak durations. However, although $n(0)$ and $w(\tau)$ cannot be considered separately, their product has a perfectly definite meaning. An expression for this product, which might be called the *unnormalized* probability density, can be obtained by differentiating (2.1):

$$n(0)w(\tau) = -\frac{dn(\tau)}{d\tau}. \tag{2.62}$$

For the special cases considered in the preceding section, we find

$$n(0)w(\tau) = \frac{w_0(b)}{2}\sqrt{\frac{\bar{K}}{2\pi}}\,\tau^{-3/2} \tag{2.63}$$

in the absence of any restoring force,

$$n(0)w(\tau) = \frac{w_0(b)}{2}\sqrt{\frac{K}{2\pi}}\,\tau^{-3/2}e^{-f_0^2\tau/2K} \tag{2.64}$$

for a constant restoring force, and

$$n(0)w(\tau) = \frac{\beta^2}{\pi}\,e^{-\beta\tau}(1-e^{-2\beta\tau})^{-3/2} \tag{2.65}$$

for a linear force $f(x) = -\beta x$ and zero level $b=0$.

Because of the absence of a normalized probability density for the peak durations, the usual characteristic function

$$\Theta(u) = \langle e^{iu\tau}\rangle, \tag{2.66}$$

satisfying the condition $\Theta(0)=1$, does not exist. Instead, we can consider the combination

$$n(0)[\Theta(u)-1],$$

which continues to have meaning in this case. In fact, multiplying (2.62) by $e^{iu\tau}-1$ and integrating by parts from 0 to ∞, we obtain

$$n(0)[\Theta(u)-1] = iu\int_0^\infty e^{iu\tau}n(\tau)\,d\tau = -\bar{n}(-iu). \tag{2.67}$$

where the expression on the right is the Carson-Laplace transform (2.28). Thus the combination $n(0)[\Theta(u)-1]$ generalizes the characteristic function, and in some cases is easier to find than the probability density itself, e.g., in the case of the linear restoring force leading to the expression (2.47).

Because of the infinite number of peaks, we can no longer talk about mean values, for example, about the mean duration $\langle\tau\rangle$ of a single peak. However, we can calculate the mean duration of the peaks *per unit time*. Although the quantities $n(0)$ and $\langle\tau\rangle$ have no meaning

separately, their product is the perfectly well-defined expression

$$n(0)\langle\tau\rangle = \int_0^\infty n(0)\,w(\tau)\,\tau\,d\tau = \frac{T_{peak}}{T_{tot}}, \tag{2.68}$$

which describes the fraction of the time during which a peak is present, i.e., the ratio of the total peak duration to the total elapsed time. Using (2.62) and (2.28), we can write (2.68) in the form

$$n(0)\langle\tau\rangle = \int_0^\infty n(\tau)\,d\tau = \frac{\bar{n}(p)}{p}\bigg|_{p=0}. \tag{2.69}$$

According to (2.67), this is just

$$n(0)\langle\tau\rangle = n(0)\,\frac{\Theta(u)-1}{iu}\bigg|_{u=0}, \tag{2.70}$$

which is the natural expression for the usual case of a normalized probability density. Using (2.67), we can also calculate higher moments of the peak durations. In fact, the usual formulas expressing the moments in terms of the characteristic function can be applied, provided we make sure that moments always appear in conjunction with the factor $n(0)$, while the characteristic function appears as the combination $n(0)[\Theta(u)-1]$. Thus, for the kth moment we have the formula

$$n(0)\langle\tau^k\rangle = \frac{1}{i^k}\frac{d^k}{du^k}n(0)[\Theta(u)-1]\bigg|_{u=0}. \tag{2.71}$$

Finally, we calculate the fraction of the time occupied by peaks of duration less than τ_1. By definition, this quantity equals

$$\frac{T_{peak}(\tau<\tau_1)}{T_{tot}} = \int_0^{\tau_1} n(0)\,w(\tau)\,\tau\,d\tau.$$

Substituting (2.62) and integrating by parts, we find that

$$\frac{T_{peak}(\tau<\tau_1)}{T_{tot}} = \int_0^{\tau_1} n(\tau)\,d\tau - \tau_1 n(\tau_1). \tag{2.72}$$

In particular, if τ_1 is much smaller than the other time constants of the system, then, according to (2.61),

$$\frac{T_{peak}(\tau < \tau_1)}{T_{tot}} = w_0(b)\sqrt{\frac{K\tau_1}{2\pi}}. \tag{2.73}$$

Thus the fraction of the time occupied by short peaks is small, even through there are infinitely many such peaks. A similar formula holds for the intervals between peaks.

4. Relation between the Distribution of Peak Durations and the Correlation Function of the Triggered Process

The statistical properties of the peaks are intimately connected with those of the process $\eta(t)$, called the *triggered process*, equal to 1 in the presence of a peak $[x(t)>b]$ and to 0 in the absence of a peak $[x(t)<b]$. The process $\eta(t)$ clearly consists of a sequence of pulses whose durations coincide with the peak durations, and can be obtained by applying a zero-memory nonlinear transformation to the original process, i.e.,

$$\eta(t) = \vartheta[x(t) - b], \tag{2.74}$$

where

$$\vartheta(z) = \begin{cases} 1 & \text{for} \quad z > 0, \\ 0 & \text{for} \quad z < 0. \end{cases}$$

In the case where $x(t)$ is a Markov process, it is easy to see (by the definition of a Markov process) that the duration of a peak, or of an interval between peaks, is statistically independent of the duration of neighboring peaks or intervals, and correspondingly, the peak and interval durations of the triggered process are independent random variables. In Chap. I.6, Sec. 5, we studied a pulse sequence of this kind, and found its spectral density. Applying formula (I.6.130) or (I.6.131), we can establish the connection between the correlation function $k_\eta(\tau)$ of the triggered process and the distribution of peak durations and intervals, described by the functions Θ_1, Θ_2 or n_1, n_2. Using

the identity

$$\frac{(1-\Theta_1)(1-\Theta_2)}{1-\Theta_1\Theta_2}=\left[\frac{1}{1-\Theta_1}+\frac{1}{1-\Theta_2}-1\right]^{-1}, \tag{2.75}$$

and noting that $\langle\varrho+\sigma\rangle^{-1}$ coincides with $n(0)$, the average total number of peaks (or intervals), we write (I.6.130) in the form

$$L\left[\frac{dk_\eta(\tau)}{d\tau};p\right] = \frac{1}{p}\left\{\frac{1}{n(0)[1-\Theta_1(ip)]}+\frac{1}{n(0)[1-\Theta_2(ip)]}-\frac{1}{n(0)}\right\}^{-1}, \tag{2.76}$$

which implies

$$pL\left[-\frac{dk_\eta}{d\tau};p\right]=\left[\frac{1}{\bar{n}_1(p)}+\frac{1}{\bar{n}_2(p)}\right]^{-1}, \tag{2.77}$$

where we have used (2.67) and the fact that $n(0)=\infty$ for an exact Markov process.[9] Formula (2.77) can be used to determine one of the functions k_η, n_1 and n_2 whenever the other two functions are known. In the special case where the level b is "symmetrically located," i.e., is such that

$$n_1(\tau)=n_2(\tau)=n(\tau),$$

(2.77) reduces to

$$pL\left[-\frac{dk_\eta}{d\tau};p\right]=\tfrac{1}{2}\bar{n}(p),$$

and hence

$$-2\frac{dk_\eta(\tau)}{d\tau}=n(\tau). \tag{2.78}$$

The correlation function $k_\eta(\tau)$ figuring in the above formulas is usually regarded as known. In particular, we can use formula (I.4.65), obtaining

$$\begin{aligned} k_\eta(\tau)&=\sum_{m=1}^{\infty}h_m^2\,e^{-\lambda_m|\tau|},\\ L\left[-\frac{dk_\eta}{d\tau};p\right]&=\sum_{m=1}^{\infty}\frac{\lambda_m h_m^2}{p+\lambda_m}, \end{aligned} \tag{2.79}$$

[9] The use of formula (I.6.130) in the case where $n(0)=\infty$ will be justified in the next section.

where

$$h_m = \int_b^\infty X_m(x)\,dx,$$

in terms of the eigenfunctions $X_m(x)$ of the Fokker-Planck equation (I.4.51).

In the case where the original process $x(t)$ is Gaussian, it is convenient to use another formula. Substituting (I.3.14) into the obvious identity

$$k_\eta(\tau) = \int_b^\infty \int_b^\infty [w(x, x_\tau) - w(x)\,w(x_\tau)]\,dx\,dx_\tau.$$

and carrying out the integration, we obtain

$$k_\eta(\tau) = \sum_{p=1}^{\infty} \left[F^{(p)}\left(\frac{b}{\sigma}\right)\right]^2 \frac{R^p}{p!}, \tag{2.80}$$

where

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\,dx.$$

It follows that

$$\frac{dk_\eta(\tau)}{d\tau} = \dot{R} \sum_{p=1}^{\infty} \left[F^{(p)}\left(\frac{b}{\sigma}\right)\right]^2 \frac{R^{p-1}}{(p-1)!},$$

where

$$\dot{R} = \frac{dR}{d\tau}.$$

Using (I.3.14) again, we find that

$$\frac{dk_\eta}{d\tau} = \dot{R}\sigma^2\,w(b,b) \tag{2.81}$$

or

$$\frac{dk_\eta}{d\tau} = \frac{1}{2\pi} \frac{\dot{R}}{\sqrt{1-R^2}}\, e^{-b^2/\sigma^2(1+R)}. \tag{2.82}$$

In the case of a Markov process, we must set $R=e^{-\beta\tau}$, but (2.82) holds for any Gaussian process, whether or not it is Markovian.

Example 1. First let $b=0$. Substituting $R=e^{-\beta\tau}$ into (2.82) and using (2.78), we find that

$$n(\tau)=-\frac{1}{\pi}\frac{\dot{R}}{\sqrt{1-R^2}}=\frac{\beta}{\pi}(e^{2\beta\tau}-1)^{-1/2}, \tag{2.83}$$

which agrees with (2.53).

Example 2. Next let $b\gg\sigma$. In the region of small values of τ, where $1-R\lesssim\sigma^2/b^2$, we can replace the expressions $1-R^2$ and $(1-R)/(1+R)$ by $2(1-R)$ and $(1-R)/2$, respectively. Then, writing (2.82) in the form

$$\frac{dk_\eta}{d\tau}=\frac{1}{\sqrt{2\pi}}\sigma w_0(b)\frac{\dot{R}}{\sqrt{1-R^2}}\exp\left\{-\frac{b^2}{2\sigma^2}\frac{1-R}{1+R}\right\}, \tag{2.84}$$

we find that

$$\frac{dk_\eta}{d\tau}=\frac{\sigma w_0(b)}{2\sqrt{\pi}}\frac{\dot{R}}{\sqrt{1-R}}e^{-b^2(1-R)/4\sigma^2}. \tag{2.85}$$

When integrated, this gives

$$k_\eta(\tau)=\frac{\sigma^2}{b}w_0(b)\operatorname{erfc}\left(\frac{b}{2\sigma}\sqrt{1-R}\right), \tag{2.86}$$

where the complementary error function is defined by (2.43). In the case of a Markov process, we can write

$$1-R=1-e^{-\beta\tau}=\beta\tau$$

in the region of small τ, and then (2.85) becomes

$$\frac{dk_\eta}{d\tau}=-\tfrac{1}{2}\sigma w_0(b)\sqrt{\frac{\beta}{\pi\tau}}e^{-b^2\beta\tau/4\sigma^2}. \tag{2.87}$$

Using a table of Laplace transforms,[10] we find that

$$pL\left[-\frac{dk_\eta}{d\tau};p\right]=w_0(b)\frac{\sigma^2\beta p}{\sqrt{4\sigma^2\beta p+b^2\beta^2}}=\tfrac{1}{2}w_0(b)\frac{pK}{\sqrt{f_0^2+2pK}}, \tag{2.88}$$

[10] See Bateman Manuscript Project, *op. cit.*, formula (3), p. 144.

which, together with the expression (2.41) for $\bar{n}_1(\rho)$, with $f_0=\beta b$, and the analogous expression for $\bar{u}_2(\rho)$, with $f_0=-\beta p$, actually satisfies equation (2.77).

5. Peaks of a Smoothed Process

Since exact Markov processes are never encountered in practice, it is of interest to show how the results obtained above can be extended to the peaks of a smoothed process. Suppose the smoothing is described by the formula

$$y(t)=\int G(t-t')x(t')\,dt', \tag{2.89}$$

where the "weight function" $G(\tau)$ has the following properties:

$$\int G(\tau)\,d\tau=1, \tag{2.90}$$

$$G(\tau)\geqslant 0, \tag{2.91}$$

$$G(\tau)=0 \quad \text{for} \quad |\tau|>\tau_{sm}\sim\int G(\tau)|\tau|\,d\tau. \tag{2.92}$$

An example of such a process is

$$G(\tau)=\begin{cases}\dfrac{1}{2\tau_{sm}} & \text{for} \quad |\tau|\leqslant\tau_{sm},\\ 0 & \text{for} \quad |\tau|>\tau_{sm}.\end{cases}$$

In the case of the exponential function

$$G(\tau)=\begin{cases}\dfrac{1}{\tau'_{sm}}\,e^{-\tau/\tau'_{sm}} & \text{for} \quad \tau>0,\\ 0 & \text{for} \quad \tau<0,\end{cases}$$

the condition (2.92) can be regarded as effectively satisfied, starting from $3\tau'_{sm}$, say, so that $G(\tau)\approx 0$ for $\tau>\tau_{sm}=3\tau'_{sm}$.

It is easy to see that a smoothing operation of the kind just described

cannot destroy a peak or an interval whose duration exceeds $2\tau_{sm}$, and accordingly, such peaks or intervals will be called "long." In fact, it follows from (2.89), (2.90) and (2.92) that

$$y(t) - b = \int G(t-t')[x(t') - b]\,dt' = \int_{t-\tau_{sm}}^{t+\tau_{sm}} G(t-t')[x(t') - b]\,dt'.$$

But the last integral is greater than zero because of (2.91), provided that $x(t')-b>0$ in the interval $t-\tau_{sm}<t'<t+\tau_{sm}$. Therefore the smoothed process $y(t)$ is sure to have a peak in the middle of a "long" peak of the unsmoothed process, and the same is true of intervals.

As far as "long" peaks are concerned, the smoothing operation can act in three ways:

1. It can change the duration of a peak (or interval) by an amount not exceeding $2\tau_{sm}$. This effect is unimportant if the duration τ is much greater than $2\tau_{sm}$.

2. It can form a "long" peak (or interval) out of "short" peaks and intervals. However, according to (2.73), the short peaks and intervals occupy a fraction of the total time equal to

$$\frac{1}{T_{tot}}[T_{peak}(\tau < 2\tau_{sm}) + T_{int}(\tau < 2\tau_{sm})] = 2w_0(b)\sqrt{\frac{K\tau_{sm}}{\pi}}.$$

If τ_{sm} is sufficiently small, this fraction is small, and hence the statistics of long peaks and intervals is only slightly changed even if all the time occupied by short peaks and intervals is converted into long peaks and intervals.

3. It can merge "long" peaks (or intervals) which are not separated by "long" intervals (or peaks), thereby forming a single peak (or interval). This effect can be examined by neglecting the first two effects and ignoring "short" peaks and intervals. As a result, we obtain the expression

$$\tilde{n}_1(p) = \frac{n_1(2\tau_{sm})\tilde{n}_1(p)}{n_1(2\tau_{sm}) + \tilde{n}_1(p)}. \tag{2.93}$$

for the distribution function $\tilde{n}_1(\tau)$ characterizing the peaks of the smoothed process, where $n_1(\tau)$ is the corresponding function for the

unsmoothed process. (There is an analogous expression for the intervals.) It follows from (3.93) that the difference between $\tilde{n}_1(\tau)$ and $n_1(\tau)$ can be neglected if $\tau_{sm} \ll \tau$.

Thus smoothing with the time constant τ_{sm} has practically no effect on the distribution of peaks of duration $\tau \gg \tau_{sm}$, and for such peaks (and intervals) we can use the results obtained earlier. The total number of peaks of the unsmoothed process is finite, of order $n_1(2\tau_{sm})$.

Applying formula (I.6.130) to the triggered process $\tilde{\eta}(t)$ with "short" peaks and intervals excluded, we find that

$$pL\left[-\frac{dk_{\tilde{\eta}}}{d\tau};\, p\right] = \left[\frac{1}{\tilde{n}_1(p)} + \frac{1}{\tilde{n}_2(p)}\right]^{-1}. \tag{2.94}$$

If we now pass to the limit $\tau_{sm} \to 0$, then $\tilde{\eta}(t)$ goes into the exact process $\eta(t)$, and (3.94) goes into (3.77). This justifies the formal application of (I.6.130) to a process with infinitely many pulses.

To investigate the behavior of the function $\tilde{n}_1(\tau)$ for $\tau \sim \tau_{sm}$, as well as for $\tau \gg \tau_{sm}$, we must take account of the concrete form of the function $G(\tau)$. It turns out that if τ_{sm} is much smaller than the other time constants of the system, then, after smoothing, the asymptotic formula (2.61) is replaced by

$$\tilde{n}(\tau) = \tfrac{1}{2} w_0(b) \sqrt{\frac{K}{\pi \tau_{sm}}}\, f\left(\frac{\tau}{2\tau_{sm}}\right), \tag{2.95}$$

where $f(z)$ is a monotonic function which is the same for all Markov processes in the diffusion approximation (see p. 1.103), depends only on the form of $G(\tau)$, and has the properties

$$f(0) \sim f(1) \sim 1, \quad f(z) \approx \frac{1}{\sqrt{z}} \quad \text{for} \quad z \gg 1. \tag{2.96}$$

Supplement

★ Since the calculations leading to the last few formulas are rather complicated, we give some of the details here:

1. After excluding peaks and intervals shorter than $2\tau_{sm}$, the peak

durations can be described by the probability density

$$w_1(\tau) = \begin{cases} -\dfrac{1}{n_1(2\tau_{sm})}\dfrac{dn_1(\tau)}{d\tau} & \text{for} \quad \tau > 2\tau_{sm}, \\ 0 & \text{for} \quad \tau \leqslant 2\tau_{sm}. \end{cases} \tag{2.97}$$

Similarly, for the intervals we have

$$w_2(\tau) = \begin{cases} -\dfrac{1}{n_2(2\tau_{sm})}\dfrac{dn_2(\tau)}{d\tau} & \text{for} \quad \tau > 2\tau_{sm}, \\ 0 & \text{for} \quad \tau \leqslant 2\tau_{sm}. \end{cases}$$

Here the mean number of peaks per unit time is $n_1(2\tau_{sm})$, while the mean number of intervals is $n_2(2\tau_{sm})$. The fraction of the total time occupied by peaks is

$$n_1(2\tau_{sm})\langle\tau_1\rangle = n_1(2\tau_{sm})\int \tau w_1(\tau)\,d\tau,$$

while that occupied by intervals is

$$n_2(2\tau_{sm})\langle\tau_2\rangle = n_2(2\tau_{sm})\int \tau w_2(\tau)\,d\tau.$$

It follows that

$$n_1(2\tau_{sm})\langle\tau_1\rangle + n_2(2\tau_{sm})\langle\tau_2\rangle = 1. \tag{2.98}$$

For sufficiently small τ_{sm}, the mean numbers of peaks and intervals are approximately the same, i.e.,

$$n_1(2\tau_{sm}) \approx n_2(2\tau_{sm}) \approx \tfrac{1}{2} w_0(b)\sqrt{\frac{K}{\pi\tau_{sm}}}, \tag{2.99}$$

and therefore (2.98) becomes

$$n_1(2\tau_{sm}) = \frac{1}{\langle\tau_1\rangle + \langle\tau_2\rangle}. \tag{2.100}$$

Using (2.97), we obtain the characteristic function

$$\Theta_1(u) = -\frac{1}{n_1(2\tau_{sm})}\int_{2\tau_{sm}}^{\infty} e^{iu\tau}\frac{dn_1(\tau)}{d\tau}\,d\tau$$

$$= e^{2iu\tau_{sm}} + \frac{iu}{n_1(2\tau_{sm})} \int_{2\tau_{sm}}^{\infty} e^{iu\tau} n_1(\tau)\, d\tau,$$

which implies

$$\Theta_1(u) \approx 1 - \frac{\bar{n}_1(-iu)}{n_1(2\tau_{sm})} \tag{2.101}$$

if $|u|$ is not too large, i.e., if $|u| \ll 1/2\tau_{sm}$.

Because of the exclusion of "short" peaks and intervals, it is no longer necessary for a peak to be followed by an interval, and in fact, a peak can be followed by another peak. The occurrence of a peak or an interval is an event which is independent of whether a peak or an interval occurred previously. Because of the (approximate) equality $n_1(2\tau_{sm}) = n_2(2\tau_{sm})$, the probability of occurrence of a peak or an interval always equals $\frac{1}{2}$. Consider a peak preceded by an interval. The peak is "solitary" (followed by an interval) with probability $\frac{1}{2}$, "double" (followed first by another peak and then by an interval) with probability $\frac{1}{4}$, "triple" with probability $\frac{1}{8}$, and so on. Therefore the probabillty density of the "merged" peak, extending from one interval to the next, is

$$\bar{w}_1(\tau) = \tfrac{1}{2} w_1(\tau) + \tfrac{1}{4} \int w_1(\tau_1) w_1(\tau - \tau_1)\, d\tau_1 + \tfrac{1}{8} \int\int w_1(\tau_1) w_1(\tau_2) w_1(\tau - \tau_1 - \tau_2)\, d\tau_1\, d\tau_2 + \ldots$$

Going over to the characteristic function, we have

$$\bar{\Theta}_1(u) = \tfrac{1}{2}\Theta_1(u) + \tfrac{1}{4}\Theta_1^2(u) + \tfrac{1}{8}\Theta_1^3(u) + \ldots = \frac{\Theta_1(u)}{2 - \Theta_1(u)}. \tag{2.102}$$

Substitution of (2.101) into (2.102) gives

$$\bar{\Theta}_1(u) = \frac{n_1(2\tau_{sm}) - \bar{n}_1(-iu)}{n_1(2\tau_{sm}) + \bar{n}_1(-iu)}. \tag{2.103}$$

Next we find the mean duration

$$\langle \tilde{\tau}_1 \rangle = \int \tau \bar{w}_1(\tau)\, d\tau = \left[\frac{1 - \Theta_1(ip)}{p} \right]_{p=0} \tag{2.104}$$

of the merged peaks, which we then compare with $\langle \tau_1 \rangle$. Substituting

(2.102) into (2.104), we obtain

$$\langle \tilde{\tau}_1 \rangle = 2 \lim_{p \to 0} \frac{1 - \Theta_1(ip)}{p[2 - \Theta_1(ip)]},$$

which implies

$$\langle \tilde{\tau}_1 \rangle = 2 \langle \tau_1 \rangle, \tag{2.105}$$

since

$$\lim_{p \to 0} \frac{1 - \Theta_1(ip)}{p} = \langle \tau_1 \rangle.$$

Applying a similar argument to the intervals, we find that

$$\tilde{\Theta}_2(u) = \frac{n_1(2\tau_{sm}) - \tilde{n}_2(-iu)}{n_1(2\tau_{sm}) + \tilde{n}_2(-iu)}$$

and

$$\langle \tilde{\tau}_2 \rangle = 2 \langle \tau_2 \rangle. \tag{2.106}$$

The mean number of merged peaks (or intervals) per unit time obviously equals

$$\tilde{n}_1(2\tau_{sm}) = \tilde{n}_2(2\tau_{sm}) = \frac{1}{\langle \tilde{\tau}_1 \rangle + \langle \tilde{\tau}_2 \rangle}.$$

Together with (2.100), (2.105) and (2.106), this implies

$$\tilde{n}_1(2\tau_{sm}) = \frac{1}{2} \frac{1}{\langle \tau_1 \rangle + \langle \tau_2 \rangle} = \frac{1}{2} n_1(2\tau_{sm}). \tag{2.107}$$

Finally, to calculate the unnormalized distribution $\tilde{n}_1(\tau)$ of the merged peaks (with respect to duration), we need only use formula (2.67), replacing the quantity $n(0)$ by the finite density of merged peaks $\tilde{n}_1(2\tau_{sm})$:

$$\tilde{n}_1(p) = \tilde{n}_1(2\tau_{sm})[1 - \tilde{\Theta}_1(ip)]. \tag{2.108}$$

Then substitution of (2.103) and (2.107) into (2.108) gives the desired formula (2.93).

2. To prove (2.94), we apply (1.6.130) to the process $\tilde{\eta}$ obtained by excluding "short" peaks and intervals. In terms of quantities pertaining to merged peaks and intervals, we have

$$pL\left[-\frac{dK_{\tilde{\eta}}}{d\tau}; p\right] = \frac{1}{\langle \tilde{\tau}_1 \rangle + \langle \tilde{\tau}_2 \rangle} \frac{[1 - \tilde{\Theta}_1(ip)][1 - \tilde{\Theta}_2(ip)]}{1 - \tilde{\Theta}_1(ip)\tilde{\Theta}_2(ip)}.$$

Just as in the derivation of (2.76), the right-hand side can be written in the form

$$\left[\frac{1}{\tilde{n}_1(2\tau_{sm})[1-\tilde{\Theta}_1(ip)]}+\frac{1}{\tilde{n}_1(2\tau_{sm})[1-\tilde{\Theta}_2(ip)]}-\frac{1}{\tilde{n}_1(2\tau_{sm})}\right]^{-1}$$

Using (2.108), we find that

$$pL\left[-\frac{dK_\theta}{d\tau};p\right]=\left[\frac{1}{\tilde{n}_1(p)}+\frac{1}{\tilde{n}_2(p)}-\frac{1}{\tilde{n}_1(2\tau_{sm})}\right]^{-1} \tag{2.109}$$

To obtain (2.94), we need only substitute (2.93) and the analogous expression for $\tilde{n}_2(p)$ into (2.109), taking account of (2.107).

3. When a Markov process is smoothed with time constant $2\tau_{sm}$, the number of peaks of duration $\tau \gg 2\tau_{sm}$ remains practically unchanged, while the number of peaks of duration $\tau \sim 2\tau_{sm}$ changes but not in order of magnitude. Therefore

$$\tilde{n}_1(2\tau_{sm}) \sim n_1(2\tau_{sm}).$$

The shorter peaks ($\tau \ll 2\tau_{sm}$) are practically all smoothed out, and hence

$$\tilde{n}_1(0) \sim \tilde{n}_1(2\tau_{sm}) \sim n_1(2\tau_{sm}). \tag{2.110}$$

The exact value of $\tilde{n}_1(0)$ depends on just how the smoothing is carried out, but as an order of magnitude estimate, (2.110) holds for any smoothing. Since

$$n_1(\tau) \approx w_0(b)\sqrt{\frac{\overline{K}}{2\pi\tau}} \tag{2.111}$$

[cf. (2.61)] if τ is appreciably smaller than the other time constants of the system (of order T_0, say), the relation (2.110) takes the form

$$\tilde{n}_1(0) \sim \tilde{n}_1(2\tau_{sm}) \sim \tfrac{1}{2} w_0(b)\sqrt{\frac{K}{\pi\tau_{sm}}}. \tag{2.112}$$

[Together, (2.99) and (2.107) serve as an example of this formula.] At the same time, as shown in Sec. 5,

$$\tilde{n}_1(\tau) \approx n_1(\tau) \quad \text{for} \quad \tau \gg 2\tau_{sm}.$$

Therefore, according to (2.111),

$$\tilde{n}_1(\tau) \approx w_0(b)\sqrt{\frac{K}{2\pi\tau}} \quad \text{for} \quad 2\tau_{sm} \ll \tau \ll T_0 . \tag{2.113}$$

Introducing an interpolation function $f(z)$, we can combine (2.112) and (2.113) into a single formula (2.95). Then it is easy to see that (2.112) and (2.113) imply the relations (2.96).★

CHAPTER 3

Smoothly Varying Noise and its Effect on Relays

In this chapter, we examine the case of smoothly varying noise, which is the opposite of the case of Markov processes considered in the preceding two chapters. Every actual process $x(t)$ is characterized by a time constant τ_{sm} describing the length of the time intervals $\Delta t \lesssim \tau_{sm}$ during which the process is smoothly varying. Our previous considerations show that $x(t)$ can be replaced by a Markov process if τ_{sm} is considerably smaller than the other time constants characterizing $x(t)$, for example τ_{cor}. However, τ_{sm} is sometimes of the same order of magnitude as τ_{cor}, e.g., when the spectral density of the noise falls off rapidly with frequency. Two spectral densities exhibiting this behavior are

$$S[x;\omega] = \frac{4\sigma^2}{\beta}\sqrt{\pi}\, e^{-\omega^2/\beta^2}$$

and

$$S[x;\omega] = \begin{cases} 2\pi\sigma^2/\beta & \text{for} \quad \omega < \beta, \\ 0 & \text{for} \quad \omega > \beta. \end{cases}$$

In fact, using the estimate $\tau_{sm} \sim x/\dot{x}$, we find that

$$\tau_{sm} = \left(\frac{\mathbf{D}x}{\mathbf{D}\dot{x}}\right)^{1/2} = \left(\frac{\int S(\omega)\, d\omega}{\int S(\omega)\,\omega^2\, d\omega}\right)^{1/2} \sim \frac{1}{\beta}$$

for each of these densities, while on the other hand, each also has $\tau_{cor} \sim 1/\beta$ [cf. (I.2.7)]. Since the spectral densities encountered in radio

engineering can quite often be approximated by such functions, the case of smoothly varying noise is of great practical interest.

The theoretical investigation of peaks of smoothly varying noise entails considerable difficulty because of the non-Markovian character of such processes. To simplify the problem, we shall assume that the threshold level $x=b$ is high. To justify this assumption, we recall from p. 20 that the operating level of devices such as pulse counters is usually much higher than the scatter of the noise fluctuations, i.e., $b-\langle x\rangle \gg \sigma(x)$, since otherwise the excessively large perturbations introduced by the noise would impede the normal operation of the device.

1. The Distribution of Peak Durations

The quantity n_0, equal to the average total number of peaks of smoothly varying noise, has already been found in Chap. 1, Sec. 1. Therefore, according to formula (2.1), to determine $n(\tau)$, the average number of peaks of duration exceeding τ, we must now find $w(\tau)$, the probability density of the distribution of peaks with respect to duration. For a smoothly varying process $x(t)$, we can write the Taylor expansion

$$x(t) = x(t_0) + \dot{x}(t_0)(t-t_0) + \tfrac{1}{2}\ddot{x}(t_0)(t-t_0)^2 + \dots, \qquad (3.1)$$

where the series converges very rapidly if $t-t_0 \ll \tau_{sm}$, in fact, approximately like a power series in $(t-t_0)/\tau_{sm}$. [As always, τ_{sm} is the time constant characterizing the rapidity of the fluctuations of $x(t)$.]

Suppose t_0 is the time of inception of a peak, so that

$$x(t_0) = b.$$

Then if the time which has elapsed since t_0 is sufficiently small, we can confine ourselves to just three terms in the expansion (3.1), obtaining

$$x(t) = b + \dot{x}(t_0)(t-t_0) + \tfrac{1}{2}\ddot{x}(t_0)(t-t_0)^2. \qquad (3.2)$$

Thus, for small intervals $t-t_0$, the curve $x(t)$ is shaped like a parabola (3.2), and can intersect the level b, thereby completing a peak, before its deviation from parabolic shape becomes appreciable. Of course, at low operating levels, such "parabolic peaks" constitute only a small

fraction of all the peaks. However, the mean duration of the peaks decreases as the operating level is increased, and at a sufficiently high level, the peak durations will as a rule become less than τ_{sm}. Then the peaks will be predominantly of parabolic shape, as described by (3.2).

If τ denotes peak duration, then

$$x(t_0 + \tau) = b,$$

which, because of (3.2), implies

$$\dot{x}\tau + \tfrac{1}{2}\ddot{x}\tau^2 = 0, \tag{3.3}$$

where

$$\dot{x} = \dot{x}(t_0), \quad \ddot{x} = \ddot{x}(t_0).$$

It follows that in the present approximation, the peak duration τ is very simply related to the first two derivatives of the process $x(t)$ at the time of inception of a peak, i.e.,

$$\tau = -2\frac{\dot{x}}{\ddot{x}}. \tag{3.4}$$

Since the initial value of $\dot{x}$ is positive, formula (3.4) is applicable only for negative values of $\ddot{x}$, and in fact, it is only in this case that we can neglect the higher-order terms in the expansion (3.1). However, it is not hard to see that the second derivative $\ddot{x}$ will be negative with overwhelming probability if the condition $b - \langle x \rangle \gg \sigma(x)$ is satisfied.

Let $w_1(\dot{x}, \ddot{x})$ be the joint probability density of the first and second derivatives of $x(t)$ at the time of inception of a peak at the level b. Then we can easily express $w_1(\dot{x}, \ddot{x})$ in terms of the three-dimensional probability density $w(x, \dot{x}, \ddot{x})$ of the coordinate x and its first two derivatives, taken at the same time. The result is

$$w_1(\dot{x}, \ddot{x}) = \begin{cases} \dfrac{w(b, \dot{x}, \ddot{x})\dot{x}}{\displaystyle\int_0^\infty \int_0^\infty w(b, \dot{x}, \ddot{x})\dot{x}\, d\dot{x}\, d\ddot{x}} & \text{for} \quad \dot{x} > 0, \\ 0 & \text{for} \quad \dot{x} < 0, \end{cases} \tag{3.5}$$

in complete analogy with (1.4). In fact, to verify (3.5), we need only repeat almost without change the argument leading to (1.4).

Given the joint distribution (3.5), it is an easy matter to find the probability density $w(\tau)$. Using (3.4), we go from the variables $\dot{x}$, $\ddot{x}$ to new variables $\dot{x}$, τ:

$$w_i(\dot{x}, \ddot{x})\, d\dot{x}\, d\ddot{x} = w_i\left(\dot{x}, -2\frac{\dot{x}}{\tau}\right) 2\frac{\dot{x}}{\tau^2}\, d\dot{x}\, d\tau.$$

Then integration with respect to $\dot{x}$ gives the final result

$$\begin{aligned} w(\tau) &= 2\int_0^\infty w_i\left(\dot{x}, -2\frac{\dot{x}}{\tau}\right)\frac{\dot{x}}{\tau^2}\, d\dot{x} \\ &= 2\int_0^\infty w_i(\tau z, -2z)\, z\, dz \qquad (\tau > 0). \end{aligned} \tag{3.6}$$

Because of the way it was derived, this formula is valid for small positive values of τ, for which the discarded terms in the series (3.1) do not play an important role. If the operating level is sufficiently high, (3.6) is valid for all the frequently encountered peaks and fails to describe only the long peaks, which occur with low probability.

To illustrate the application of formula (3.6), we now consider a Gaussian random process $x(t)$, with mean zero and correlation function

$$\langle x x_\tau \rangle = \sigma^2 R(\tau). \tag{3.7}$$

Because of the assumption that $x(t)$ is smoothly varying, the correlation coefficient has the series expansion

$$R(\tau) = 1 - \frac{1}{2} R_2 \tau^2 + \frac{1}{4!} R_4 \tau^4 - \ldots \tag{3.8}$$

$$R_{2k} = (-1)^k \frac{d^{2k} R(0)}{d\tau^{2k}}, \tag{3.9}$$

which converges rapidly for values of τ such that the higher-order terms in (3.1) can be neglected. To find the three-dimensional probability density $w(x, \dot{x}, \ddot{x})$ figuring in formula (3.5), we first calculate the

correlation matrix

$$\begin{Vmatrix} \langle x^2\rangle & \langle x\dot{x}\rangle & \langle x\ddot{x}\rangle \\ \langle \dot{x}x\rangle & \langle \dot{x}^2\rangle & \langle \dot{x}\ddot{x}\rangle \\ \langle \ddot{x}x\rangle & \langle \ddot{x}\dot{x}\rangle & \langle \ddot{x}^2\rangle \end{Vmatrix} = \sigma^2 \begin{Vmatrix} 1 & 0 & -R_2 \\ 0 & R_2 & 0 \\ -R_2 & 0 & R_4 \end{Vmatrix}, \tag{3.10}$$

where the mean values $\langle x\dot{x}\rangle$, $\langle \dot{x}\ddot{x}\rangle$, etc., are easily obtained by differentiating the correlation function (3.7) and then setting $\tau=0$. It is apparent that the first derivative $\dot{x}$ is not correlated with either x or $\ddot{x}$, and hence is independent of x and $\ddot{x}$, since $x(t)$ is Gaussian. Correspondingly, the three-dimensional probability density can be written as a product

$$w(x,\dot{x},\ddot{x}) = w(\dot{x})\,w(x,\ddot{x}). \tag{3.11}$$

Using the correlation matrix (3.10) and the familiar formula for the multidimensional probability density of a Gaussian process in terms of the correlation matrix [see Chap. 1.3, esp. formula (1.3.3)], we find at once that

$$w(\dot{x}) = \frac{1}{\sqrt{2\pi\sigma^2 R_2}}\, e^{-\dot{x}^2/2\sigma^2 R_2}, \tag{3.12}$$

$$w(x,\ddot{x}) = \frac{1}{2\pi\sigma^2\sqrt{R_4 - R_2^2}} \exp\left\{-\frac{R_4 x^2 + 2R_2 x\ddot{x} + \ddot{x}^2}{2\sigma^2(R_4 - R_2^2)}\right\}. \tag{3.13}$$

In the same way, by comparing (3.5), (3.11) and (1.4), we see that the probability density $w_i(\dot{x},\ddot{x})$ of the initial values of $\dot{x}$ and $\ddot{x}$ factors into a product

$$w_i(\dot{x},\ddot{x}) = w_i(\dot{x})\,w(\ddot{x}|b), \tag{3.14}$$

where

$$w_i(\dot{x}) = \frac{\dot{x}}{\sigma^2 R_2}\, e^{-\dot{x}^2/2\sigma^2 R_2} \qquad (\dot{x} > 0) \tag{3.15}$$

is the probability density of the first derivative $\dot{x}$ at the time of inception of a peak [cf. (1.4), (1.7), (1.8)], while

$$w(\ddot{x}|b) = \frac{w(b,\ddot{x})}{w(b)} = \frac{1}{\sqrt{2\pi\sigma^2(R_4 - R_2^2)}} \exp\left\{-\frac{1}{2\sigma^2}\,\frac{(\ddot{x} + R_2 b)^2}{R_4 - R_2^2}\right\} \tag{3.16}$$

is the conditional probability density of the second derivative $\ddot{x}$ at a

given level b. Integrating (3.6) with respect to τ and using (3.15), we obtain the distribution of the peak durations in the form

$$F(\tau) = \int_0^{\tau} w(\tau)\,d\tau = 2\int_0^{\infty} (1 - e^{-\tau^2 z^2/2\sigma^2 R_2})\, w(-2z|b)\,dz. \tag{3.17}$$

Substituting (3.16) into (3.17), we use the formula

$$\int_0^{\infty} e^{-p^2z^2+2qz}\,dz = \frac{\sqrt{\pi}}{2p} e^{q^2/p^2} \operatorname{erfc}(-q/p) \tag{3.18}$$

[cf. (2.43)], where

$$q = \frac{R_2 b}{\sigma^2(R_4 - R_2^2)}, \tag{3.19}$$

first setting

$$p = \frac{2}{\sigma^2(R_4 - R_2^2)}$$

and then

$$p = \frac{1}{2\sigma^2}\left(\frac{4}{R_4 - R_2^2} + \frac{\tau^2}{R_2}\right).$$

The result is

$$\begin{aligned} 2F(\tau) = \operatorname{erfc}\left\{-\frac{b}{\sqrt{2\gamma}}\right\} - \left(1 + \frac{\gamma}{4}R_2\tau^2\right)^{-1/2} \\ \times \exp\left\{-\frac{b^2}{2\sigma^2}\frac{R_2\tau^2}{4 + \gamma R_2\tau^2}\right\} \operatorname{erfc}\left\{-\frac{b}{\sigma\sqrt{2\gamma}}\left(1 + \frac{\gamma}{4}R_2\tau^2\right)^{-1/2}\right\}, \end{aligned} \tag{3.20}$$

where

$$\gamma = \frac{R_4 - R_2^2}{R_2^2}. \tag{3.21}$$

Formula (3.21) is valid for values of τ which are small enough to allow us to use (3.2) instead of (3.1), dropping all terms of order 3 or higher. The rate of convergence of the series (3.1) depends on that of the

series (3.8). Therefore the condition for applicability of the approximation (3.2) can be written in the form

$$R_2\tau^2 \ll 1, \quad \gamma R_2\tau^2 \ll 1. \tag{3.22}$$

These inequalities can be used to deduce simplified (but less accurate) expressions from (3.22). Thus, taking account of only the first τ-dependent term in $[1+(\gamma R_2\tau^2/4)]^{-1}$, i.e., writing

$$\begin{aligned} 4(4+\gamma R_2\tau^2)^{-1} &\approx 1 - \tfrac{1}{4}\gamma R_2\tau^2, \\ 2(4+\gamma R_2\tau^2)^{-1/2} &\approx 1 - \tfrac{1}{8}\gamma R_2\tau^2 \end{aligned} \tag{3.23}$$

in (3.20), we obtain

$$\begin{aligned} 2F(\tau) = \operatorname{erfc}\left\{-\frac{b}{\sigma\sqrt{2\gamma}}\right\} - \left(1 - \frac{\gamma}{8}R_2\tau^2\right) \\ \times \exp\left\{-\frac{b^2}{8\sigma^2}R_2\tau^2\right\} \operatorname{erfc}\left\{-\frac{b}{\sigma\sqrt{2\gamma}}\left(1 - \frac{\gamma}{8}R_2\tau^2\right)\right\}. \end{aligned} \tag{3.24}$$

At low operating levels, the width of the region of applicability of the results obtained above is only a small fraction of the mean peak duration $\langle\tau\rangle$, and hence these results can be used only to calculate the density of the rare short peaks. As the operating level is increased, the mean peak duration becomes smaller and falls into the region of applicability of our formulas, which can now be used to calculate the rate of occurrence of almost all the peaks, except the very rare long peaks. In other words, in this case our results determine the shape of almost the whole distribution $w(\tau)$, except for an unimportant "tail" corresponding to small values of $w(\tau)$.

Suppose the condition

$$b^2 \gg \sigma^2, \tag{3.25}$$

characterizing a high operating level, is satisfied. Then (3.24) gives a Rayleigh distribution for the peak durations, i.e.,

$$\begin{aligned} F(\tau) &= 1 - \exp\left\{-\frac{b^2}{8\sigma^2}R_2\tau^2\right\}, \\ w(\tau) &= \frac{b^2R_2}{4\sigma^2}\tau \exp\left\{-\frac{b^2}{8\sigma^2}R_2\tau^2\right\}. \end{aligned} \tag{3.26}$$

Correspondingly, the most probable value of τ equals $2\sigma/b\sqrt{R_2}$, while the mean peak duration is

$$\langle\tau\rangle = \sqrt{\frac{\pi}{2}}\,\frac{2\sigma}{b\sqrt{R_2}}. \tag{3.27}$$

Using the expression (1.9) for the average total frequency of occurrence of peaks, we find that the number of peaks of duration exceeding τ is given by

$$n(\tau) = n_0[1 - F(\tau)] = \frac{\sqrt{R_2}}{2\pi}\exp\left\{-\frac{b^2}{2\sigma^2}\left(1 + \frac{R_2\tau^2}{4}\right)\right\}. \tag{3.28}$$

There is another, easier way of deriving (3.26): According to (3.16), the conditional mean square deviation

$$\sigma(\ddot{x}) = \left[\int(\ddot{x} + R_2 b)^2\, w(\ddot{x}|b)\, d\ddot{x}\right]^{1/2} = \sigma\sqrt{R_4 - R_2^2} \tag{3.29}$$

is much less than the conditional mean value $-R_2 b$, because of (3.25). Therefore the second derivative $\ddot{x}$ at the time of inception of a peak can be regarded as nonrandom and equal to $-R_2 b$. Thus the peak duration (3.4) becomes

$$\tau = \frac{2}{R_2 b}\,\dot{x}, \tag{3.30}$$

and is determined entirely by the size of the first derivative $\dot{x}$, which has the distribution (3.15). Then, making the simple change of variables (3.29) in (3.15), i.e., writing

$$w(\tau)\, d\tau = w_l\left(\frac{b}{2}R_2\tau\right)\frac{b}{2}R_2\, d\tau, \tag{3.31}$$

we immediately arrive at formula (3.26).

Finally, it should be noted that the above method for finding the distribution of peak durations is applicable not only in the case of Gaussian noise, but also in any other case where the three-dimensional probability density $w(x, \dot{x}, \ddot{x})$ can be calculated.

2. Other Methods for Investigating Peaks of Smoothly Varying Noise

2.1. Another way of finding the distribution of peak durations starts from a calculation of the correlation function (2.74) of the triggered process. In the case of slowly varying noise, the durations of the peaks and intervals are correlated random variables. Therefore a formula like (I.6.130) is not valid for low and moderate operating levels, and the results obtained by using (I.6.130) can only serve as a rough guide. However, at high operating levels, the mean interval length between peaks is much greater than τ_{sm} or τ_{cor}, and the correlations disappear, allowing us to use (I.6.130) or (2.76).

Thus, when the intervals between peaks are much greater than τ_{cor} (on the average), we can neglect correlations. Then it is a good approximation to regard the initial times of the peaks as having a Poisson distribution (cf. p. I.144). Therefore the intervals have the probability density

$$w_2(\tau) = n_0 e^{-n_0\tau} \tag{3.32}$$

[cf. (I.6.34)], and hence

$$\Theta_2(ip) = \frac{n_0}{p + n_0}.$$

Substituting this expression into (2.76), we find that

$$pL\left[-\frac{dk_\eta}{d\tau}; p\right] = n_0\left[\frac{1}{1 - \Theta_1(ip)} + \frac{n_0}{p}\right]^{-1} \tag{3.33}$$

Because of the inequality

$$\langle\tau\rangle \ll \frac{1}{n_0}, \tag{3.34}$$

which means that the mean peak duration is much smaller than the mean interval duration, the term n_0/p in (3.33) is much smaller than the term

$$\frac{1}{1 - \Theta_1(ip)} = \frac{1}{\langle\tau\rangle p + \ldots}$$

if $p \sim 1/\langle\tau\rangle$, and can therefore be neglected. Then (3.33) becomes

$$pL\left[-\frac{dk_\eta}{d\tau}; p\right] = n_0[1 - \Theta_1(ip)] = \bar{n}(p)$$

[cf. (2.67)], or

$$-\frac{dk_\eta(\tau)}{d\tau} = n(\tau) = n_0 \int_\tau^\infty w(\tau)\,d\tau. \tag{3.35}$$

This formula differs from (2.78) only by the absence of the factor of 2, and can be written in the form

$$w(\tau) = \frac{1}{n_0}\frac{d^2k_\eta(\tau)}{d\tau^2}. \tag{3.36}$$

For example, suppose $x(t)$ is a Gaussian process. Then according to (3.35) and (2.84),

$$n(\tau) = -\frac{1}{\sqrt{2\pi}}\sigma w_0(b)\frac{\dot{R}}{\sqrt{1-R^2}}\exp\left\{-\frac{b^2}{2\sigma^2}\frac{1-R}{1+R}\right\}, \tag{3.37}$$

and hence

$$w(\tau) = \frac{1}{\sqrt{R_2}}\frac{d}{d\tau}\left[\frac{R}{\sqrt{1-R^2}}\exp\left\{-\frac{b^2}{2\sigma^2}\frac{1-R}{1+R}\right\}\right] \tag{3.38}$$

[cf. (1.9)]. If $b \gg \sigma$, $1 - R \ll 1$, we can replace the correlation coefficient $R(\tau)$ by $1 - \frac{1}{2}R_2\tau^2$. Then formulas (3.37) and (3.38) reduce to (3.28) and (3.26), respectively.

2.2. A general solution of the problem of calculating the distribution of peak durations can be found by using the theory of correlated random points developed in Chap. I.6 (esp. Secs. 1 and 4), which leads to an expression in the form of an infinite series. In practice, this series can be used only to obtain approximate results, since the amount of calculation required to achieve a given accuracy grows rapidly as the accuracy is increased. However, in some cases, e.g., for moderate operating levels and smoothly varying noise, this is the only suitable method.

Suppose the process $x(t)$ has a peak beginning at the time $t = t_0 = 0$, so that

$$x(0) = b, \quad \dot{x}(0) > 0.$$

Subsequently, the curve $x(t)$ may intersect the threshold level several times. Suppose $x(t)$ makes *downward* crossings of the level $x = b$ (going from the region above $x = b$ to the region below $x = b$) at the points t_j, so that

$$x(t_j) = b\,, \quad \dot{x}(t_j) < 0\,. \tag{3.39}$$

Then the points t_j form a system of identical random points, described by the distribution functions

$$f_1(t_1), f_2(t_1, t_2), \ldots, f_r(t_1, \ldots, t_r), \ldots$$

Using the definition (I.6.1) of these functions, and following the derivation of formula (1.3), we find that

$$f_r(t_1, \ldots, t_r) = \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} w_{cond}(b, \dot{x}_1, \ldots, b, \dot{x}_r) |\dot{x}_1| \ldots |\dot{x}_r|\, d\dot{x}_1 \ldots d\dot{x}_r\,, \tag{3.40}$$

where

$$w_{cond}(x_1, \dot{x}_1, \ldots, x_r, \dot{x}_r) = \frac{\int_0^\infty w_{r+1}(b, \dot{x}_0, x_1, \dot{x}_1, \ldots, x_r, \dot{x}_r)\, \dot{x}_0 d\dot{x}_0}{\int_0^\infty w_1(b, \dot{x}_0)\, \dot{x}_0 d\dot{x}_0} \tag{3.41}$$

and

$$w_{r+1}(x_0, \dot{x}_0, \ldots, x_r, \dot{x}_r) = w[x(0), \dot{x}(0), \ldots, x(t_r), \dot{x}(t_r)]\,.$$

The system of points (3.39), defined for $t > 0$, is nonstationary even when $x(t)$ itself is stationary, and in fact belongs to the category of "switched-on" processes (see p. I.132). However, for $t \gg \tau_{corr}$, (3.39) becomes a stationary system (see p. I.147). For such values of t, the conditional probability density (3.41) coincides with the unconditional probability density, and the functions $f_r(t_1, \ldots, t_r)$ depend only on the time differences. In particular, $f_1(t_1)$ reduces to a constant

$$f_1 = n_0 \tag{3.42}$$

and $f_2(t_1, t_2)$ to the function

$$f_2(t_1 - t_2) = \int_{-\infty}^{0} \int_{-\infty}^{0} w_2(b, \dot{x}_1, b, \dot{x}_2) |\dot{x}_1|\, |\dot{x}_2|\, d\dot{x}_1\, d\dot{x}_2 \tag{3.43}$$

depending only on $t_1 - t_2$.

The duration τ of a peak is the distance from the origin to the "first" random point t_1, i.e., the random point nearest the origin. If the peak duration τ exceeds τ_m, then no point at all falls in the interval $0 \leqslant t \leqslant \tau_m$. The probability of this event is

$$\mathbf{P}\{\tau > \tau_m\} = \int_{\tau_m}^{\infty} w(\tau)\, d\tau, \tag{3.44}$$

which, according to (I.6.13), can be written in terms of the generating functional as

$$\mathbf{P}\{\tau > \tau_m\} = L_{\tau_m}[-1], \tag{3.45}$$

where

$$L_\tau[-1] = 1 + \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \int_0^\tau \dots \int_0^\tau f_r(t_1, \dots, t_r)\, dt_1 \dots dt_r, \tag{3.46}$$

by (I.6.4). The last three formulas, together with the symmetry of $f_r(t_1, \dots, t_r)$, imply

$$w(\tau) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(r-1)!} \int_0^\tau \dots \int_0^\tau f_r(\tau, t_1, \dots, t_{r-1})\, dt_1 \dots dt_{r-1}. \tag{3.47}$$

The probability density $w(\tau)$ can also be expressed in terms of the correlation functions $g_r(t_1, \dots, t_r)$, related to the distribution functions by the formulas (I.6.7). In fact, it follows from (3.44), (3.45) and (I.6.9) that

$$\int_\tau^\infty w(\tau)\, d\tau = \exp\left\{ -\int_0^\tau f_1(\tau)\, d\tau + \sum_{r=2}^{\infty} \frac{(-1)^r}{r!} \int_0^\tau \dots \int_0^\tau g_r(t_1, \dots, t_r)\, dt_1 \dots dt_r \right\}. \tag{3.48}$$

In the case of smoothly varying noise, the derivative $\dot{x}(t)$ cannot change sign rapidly. Since the time it takes $\dot{x}(t)$ to change appreciably is of order τ_{cor}, there is only a very small probability of two neighboring intersections having a separation $\tau \ll \tau_{cor}$. Therefore the distribution functions $f_2(t_1, t_2), f_3(t_1, t_2, t_3), \dots$ take small values at neighboring instants of time, i.e.,

$$f_r(t_1, \dots, t_r) \ll f_1(t_1) \dots f_1(t_r) \quad (r = 2, 3, \dots) \tag{3.49}$$

if $|t_i - t_j| \ll \tau_{cor}$ for all i, j. Consequently, if $\tau \ll \tau_{cor}$, the only important term in (3.47) is

$$w(\tau) = f_1(\tau). \tag{3.50}$$

Thus, if the operating level b is high, so that the mean peak duration is very much less than τ_{cor}, formula (3.50) gives a satisfactory solution of the problem, since then it correctly describes the behavior of $w(\tau)$ for the frequently occurring peaks. The formula breaks down in the region $\tau \sim \tau_{cor}$, corresponding to the infrequently occurring peaks. It should be noted that according to (3.40) and (3.41), in order to obtain the distribution (3.50), we have to calculate the integral

$$n_0 f_1(\tau) = \int_0^\infty \int_{-\infty}^0 w_2(b, \dot{x}_0, b, \dot{x}_\tau)\, \dot{x}_0 |\dot{x}_\tau|\, d\dot{x}_0\, d\dot{x}_\tau. \tag{3.51}$$

where $\dot{x} = \dot{x}(0)$, $\dot{x}_\tau = \dot{x}(\tau)$.

★In the case of a low or moderate operating level, it is important to know the probability density $w(\tau)$ in the region $\tau \sim \tau_{cor}$. Study of this region entails calculation of higher-order terms in the series (3.47) or (3.48) and evaluation of multiple integrals of the form (3.40), and hence leads to technical difficulties. These difficulties can be avoided to a certain extent by the special device of expressing the higher-order correlation functions $g_r(t_1, \ldots, t_r)$ in terms of lower-order correlation functions. In so doing, we assume that the higher-order correlations are characterized by the same correlation time as lower-order correlations, and we express all the correlation functions in terms of the first two, i.e., $f_1(t)$ and $g_2(t_1, t_2)$. At the same time, the actual system of random points is replaced by a system of the kind considered in Chap. 6, Sec. 4. To illustrate the method, we solve the problem of determining $w(\tau)$ in the approximation corresponding to a system of nonapproaching points (see p. I.160), making use of the approximation (3.49) which states that the probability of points being separated by small time intervals is small.

Thus we begin by setting $t_1 = \ldots = t_r$, obtaining the relations

$$f_r(t_1, \ldots, t_1) = 0 \qquad (r = 2, 3, \ldots), \tag{3.52}$$

which allows us to determine $g_r(t_1, \ldots, t_1)$. In fact, substitution of $t_1 = \ldots = t_r$, $z_1 = \ldots = z_r$ in (I.6.6) gives

$$\sum_{r=1}^{\infty} \frac{1}{r!} g_r(t_1, \ldots, t_1) z_1^r = \ln[1 + f_1(t_1) z_1]. \tag{3.53}$$

Then, expanding the logarithm in series

$$\ln(1 + f_1 z_1) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} f_1^r z_1^r,$$

and equating coefficients of like powers of z_1 in (3.53), we find that

$$g_r(t_1, \ldots, t_1) = (-1)^{r-1}(r-1)! f_1^r(t_1). \tag{3.54}$$

Similarly, suppose we choose distinct but neighboring time instants $t_1, t_2 \ldots$ (where $|t_i - t_j| \ll \tau_{cor}$) and distinct z_i in formula (I.6.6), and then neglect the terms in $f_2, f_3, \ldots$ as before, using (3.49). This gives

$$g_r(t_1, \ldots, t_r) = (-1)^{r-1}(r-1)! f_1(t_1) \ldots f_1(t_r) \quad (r = 2, 3, \ldots), \tag{3.55}$$

where $|t_i - t_j| \ll \tau_{cor}$ for all i, j.

These relations are good approximations if the correlation is strong. In fact, let $R(t_1, t_2)$ be the correlation coefficient of the original process $x(t)$. Then for coincident times $t_1 = \ldots = t_r$, when the correlation is total and R becomes unity, formula (3.54) holds exactly. For $|t_i - t_j| \ll \tau_{cor}$, we have $R(t_i, t_j) \approx 1$, and the relations (3.55) hold approximately. As the intervals between points are increased until they are of order τ_{cor}, the correlations which only extend over intervals $|t_i - t_j| \sim \tau_{cor}$ begin to disappear, and correspondingly, the functions $g_r(t_1, \ldots, t_r)$ describing these correlations, and the function $R(t_i, t_j)$, grow smaller. If at least one of the differences $|t_i - t_j|$ is much greater than τ_{cor} and if the corresponding value $R(t_i, t_j)$ is approximately zero, then the correlation function $g_r(t_1, \ldots, t_r)$ will be approximately zero. This suggests approximating $g_r(t_1, \ldots, t_r)$ by an expression containing the correlation coefficient R. One such representation is

$$g_r(t_1, \ldots, t_r) = (-1)^{r-1}(r-1)! f_1(t_1) \ldots f_1(t_r) \{R(t_1, t_2) \ldots R(t_1, t_r)\}_s, \tag{3.56}$$

where, as on p. I.19, the symbol $\{\ldots\}_s$ denotes the operation of

symmetrizing the expression in brackets with respect to all its arguments, and is introduced to make $g_r(t_1, \ldots, t_r)$ satisfy the necessary symmetry conditions. If the correlation coefficient is smoothly varying and satisfies the conditions

$$R(t_1, t_1) = 1, \tag{3.57}$$

$$\int |R(t_1, t_2)|\, dt_2 \sim \tau_{cor}, \tag{3.58}$$

then the correlation functions (3.56) satisfy the conditions (3.54) and (3.55), and moreover vanish, as required, if at least one difference $|t_i - t_j| \gg \tau_{cor}$.

Setting $r=2$ in (3.56), we obtain the formula

$$g_2(t_1, t_2) = -f_1(t_1) f_1(t_2) R(t_1, t_2), \tag{3.59}$$

which we use to *define* the correlation coefficient $R(t_1, t_2)$. Then the other relations (3.56) will be approximations for $r>2$. Using (I.6.7), we can write (3.59) in the form

$$R(t_1, t_2) = 1 - \frac{f_2(t_1, t_2)}{f_1(t_1) f_1(t_2)}. \tag{3.60}$$

It is easy to see that the condition (3.57) is satisfied because of (3.52), while the condition that R be smoothly varying guarantees that the process $x(t)$ is smoothly varying. The condition (3.58) is merely a definition of the correlation time, which, for subsequent purposes, it is convenient to replace by

$$\lim_{t_1 \to \infty} \int_0^\infty R(t_1, t_2)\, dt_2 = \tau_{cor}, \tag{3.61}$$

instead of the approximate formula (3.38).

To find the distribution $w(\tau)$, we must still substitute (3.56) into (3.48), or else use (I.6.64) which gives

$$\int_\tau^\infty w(\tau)\, d\tau = e^{-S(t)}, \tag{3.62}$$

where

$$S(\tau) = -\int_0^\tau \frac{\ln\left[1 - \int_0^t R(t,t')f_1(t')\,dt'\right]}{\int_0^t R(t,t')f_1(t')\,dt'} f_1(t)\,dt. \qquad (3.63)$$

These formulas can be used both for small and for large values of τ. If $\tau \ll \tau_{cor}$, we can set $R=1$ in (3.63), which allows us to bring the integrand, except for the factor $f_1(t)$, in front of the integral sign. This gives

$$\int_\tau^\infty w(\tau)\,d\tau = 1 - \int_0^\tau f_1(t)\,dt, \qquad (3.64)$$

which is equivalent to (3.50). In the opposite case, where $\tau \gg \tau_{cor}$, formula (3.63) can again be drastically simplified. For values of t such that

$$t \gg \tau_{cor}, \quad \tau - t \gg \tau_{cor}$$

(i.e., which are far from both the beginning and the end of a peak), we have

$$\int_0^\tau R(t,t')f_1(t')\,dt' \approx \int_0^\infty R(t,t')n_0\,dt' = n_0\tau_{cor},$$

because of (3.42) and (3.61). Therefore (3.62) and (3.63) imply

$$\int_\tau^\infty w(\tau)\,d\tau = \text{const}\cdot e^{\ln(1-n_0\tau_{cor})\tau/\tau_{cor}} = \text{const}\cdot(1-n_0\tau_{cor})^{\tau/\tau_{cor}}. \qquad (3.65)$$

For intermediate values $\tau \sim \tau_{cor}$, there is a gradual transition from the behavior described by (3.64) to that described by (3.65).

According to (3.40) and (3.60), calculation of the correlation coefficient $R(t_1, t_2)$ entails evaluation of a triple integral involving the probability density $w_3(x_0, \dot{x}_0, x_1, \dot{x}_1, x_2, \dot{x}_2)$. This can be avoided at the cost of diminishing the accuracy of the result, by replacing $f_2(t_1, t_2)$ in (3.60) by the limit function (3.43). Then

$$R(t_1,t_2) = R(t_1-t_2) = 1 - \frac{1}{n_0^2}\int_{-\infty}^0\int_{-\infty}^0 w_2(b,\dot{x}_1,b,\dot{x}_2)|\dot{x}_1|\,|\dot{x}_2|\,d\dot{x}_1\,d\dot{x}_2,$$

and the calculation of $R(\tau)$ is of the same order of difficulty as that of $f_1(\tau)$ [cf. (3.51)].★

2.3. As we have just seen, although it is very difficult to calculate the exact distribution of peak durations, several approximate formulas, of varying degrees of complexity, are available for the case of high thresholds $b \gg \sigma$. As the threshold is lowered, these formulas all break down, but some remain better approximations than others. To illustrate the extent to which the theoretical formulas continue to apply when the ratio b/σ is not very large, we now consider some experimental data. The distribution of peak durations for a Gaussian random process with spectral density

$$S[x;\omega] = \begin{cases} \dfrac{1}{2\pi}\dfrac{\sigma^2}{\omega_2 - \omega_1} & \text{for} \quad \omega_1 < |\omega| < \omega_2, \\ 0 & \text{otherwise} \end{cases}$$

$$\left(\frac{\omega_1}{2\pi} = f_1 = 440 \text{ kc/sec}, \quad \frac{\omega_2}{2\pi} = f_2 = 10{,}250 \text{ kc/sec}\right)$$

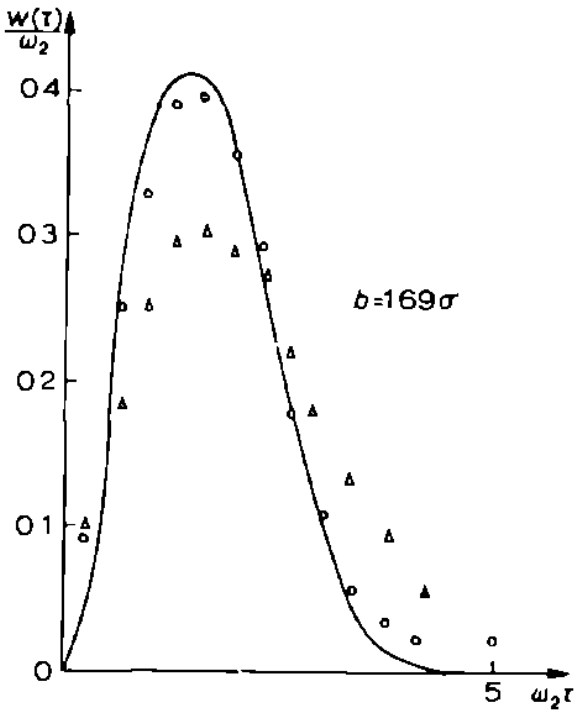

Fig. 4. Theoretical and experimental distribution of peak durations for $b = 1.69\sigma$.

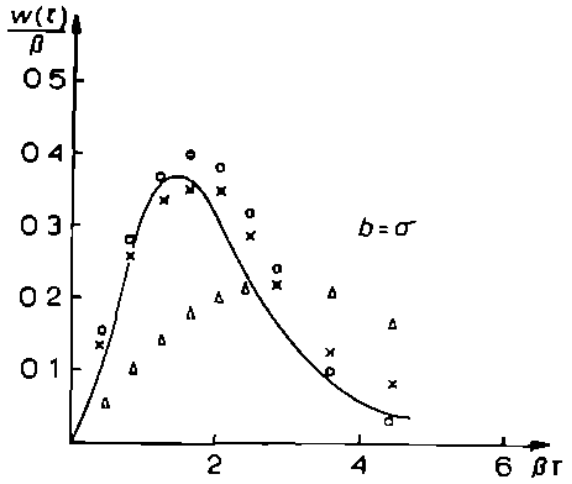

Fig. 5. Theoretical and experimental distribution of peak durations for $b = \sigma$.

and threshold $b = 1.69$ is shown in Figure 4, while the distribution for a Gaussian process with correlation function $\sigma^2 e^{-\beta^2 \tau^2/4}$ and threshold $b = \sigma$ is shown in Figure 5. In each case, the solid curve represents the experimental data.[1]

The simplest theoretical formula (3.26) gives a Rayleigh distribution, and the corresponding points are indicated in the figures by triangles. This formula is not as accurate as formula (3.38), which gives rise to the points indicated by circles.[2] Here the agreement between theory and experiment is satisfactory even for $b = \sigma$. Finally, the curves in Figure 5 indicate the results of calculations based on formulas (3.50) and (3.51). These calculations, which are more tedious than the others, also lead to satisfactory agreement with experiment.

3. The Area under the Peaks

The operating level b has so far been defined as the minimum value

[1] See V. I. Tikhonov and I. N. Amiantov, *Probability densities for the duration of noise peaks*, Radiotekh. 15, 10 (1960). An English translation appears in the volume "Non-Linear Transformations of Stochastic Processes," edited by P. I. Kuznetsov, R. L. Stratonovich and V. I. Tikhonov, Pergamon Press, Inc., New York (1965), p. 382.

[2] The derivation of (3.38) makes essential use of the assumption that the peak and interval durations are independent random variables.

of the "input coordinate" $x(t)$ needed to activate the relay. If $x(t)$ is sufficiently slowly varying, then the relay will actually operate every time $x(t)$ first exceeds the level b. However, departures from this rule are observed in practice, because of the finite, nonnegligible rate of change of $x(t)$, i.e., some peaks fail to activate the relay and as a result "go unnoticed." The assumption made on p. 21 to the effect that the relay operates if and only if the peak duration exceeds a certain length τ is not always a satisfactory approximation to the true state of affairs.

Special studies of the effect on electronic relays of pulses of various shapes support the conclusion that the following rule is the most reliable: There exists a level b and a minimum area S_m, which depend only slightly on the form of the pulse, such that the relay operates every time the area "under the pulse (peak)" exceeds S_m. Here we have in mind the area bounded above by the curve $x(t)$ and below by the line $x=b$, as shown in Figure 6. The calculation of the area is assumed to

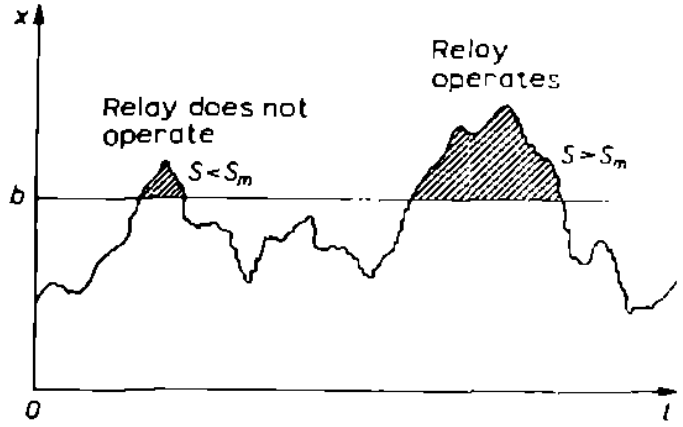

Fig. 6. The area under peaks of random noise.

begin anew every time $x(t)$ makes an upward crossing of the level b. Of course, this condition for the functioning of the relay is not exact, but it does adequately describe the operating characteristics of a wide class of relays, especially in the case where the input signal has a tendency to make well-separated crossings of the threshold.

The requirement of a minimum peak area rather than a minimum peak duration takes account properly of the fact that even a very short pulse can cause the relay to operate if the pulse is very strong. In other

words, the minimum area requirement takes account not only of the peak duration but also of the extent to which the peak exceeds the threshold, and hence can be regarded as a requirement imposed on the "energy" of the peaks.

As before, let τ denote the peak duration, starting from the time $t=t_0$. Then the operating condition just introduced takes the form

$$S \equiv \int_{t_0}^{t_0+\tau} [x(t)-b]\,dt > S_m, \tag{3.66}$$

where the quantity S will be called the *excess area*. The mean number of omitted or "skipped" peaks can be simply expressed in terms of the distribution of the peaks with respect to the excess area:

$$n_{skip} = n_0 \int_0^{S_m} w(S)\,dS \equiv n_0 F(S_m). \tag{3.67}$$

Since the total number of peaks n_0 has already been found [recall (1.5) and (1.9)], it only remains to calculate the quantity $w(S)$. In so doing, we assume, as before, that $x(t)$ is smoothly varying and that the operating level is sufficiently high.

As noted in Sec. 1, the peaks of sufficiently short duration are of parabolic shape, and are described by the first three terms of the Taylor series expansion of $x(t)$:

$$x(t) = x(t_0) + \dot{x}(t_0)(t-t_0) + \tfrac{1}{2}\ddot{x}(t_0)(t-t_0)^2. \tag{3.68}$$

Here the duration of the peak equals

$$\tau = -2\frac{\dot{x}}{\ddot{x}},$$

where $\dot{x}=\dot{x}(t_0)$, $\ddot{x}=\ddot{x}(t_0)$ [cf. (3.4)], while the excess area is given by the integral

$$S = \int_0^{\tau}\left(\dot{x}s + \frac{1}{2}\ddot{x}s^2\right)ds = \dot{x}\frac{\tau^2}{2} + \ddot{x}\frac{\tau^3}{6} = \frac{2}{3}\frac{\dot{x}^3}{\ddot{x}^2}. \tag{3.69}$$

As before, we use the expression (3.5) for $w_1(\dot{x}, \ddot{x})$, the probability density of the first two derivatives at the time of inception of a peak. This leads to the following expression for the probability density of the excess area:

$$w(S) = (\tfrac{3}{2})^{1/3} S^{-2/3} \int_0^\infty w_1[(\tfrac{3}{2}S)^{1/3} z^2, z^3] z^4 \, dz. \tag{3.70}$$

Formula (3.70) enables us to calculate the mean number of skipped peaks (3.67), even in the case of low operating levels, provided that the minimum area is sufficiently small, i.e., $S_m \ll \tau_{sm}^2 \sigma(\dot{x})$. However, for high operating levels, the evaluation of $w(S)$ can be greatly simplified. For a Gaussian random process, with mean zero and correlation function (3.7), the second derivative at a high threshold $x = b$ is practically nonrandom, and equals $\ddot{x} = -R_2 b$ (see p. 58). Therefore the excess area (3.69) is determined entirely by the size of the first derivative:

$$S = \frac{2}{3} \frac{\dot{x}^3}{R_2^2 b^2}. \tag{3.71}$$

According to (3.15), $\dot{x}$ has the distribution function

$$F(\dot{x}) = 1 - \exp\left\{-\frac{\dot{x}^2}{2\sigma^2 R_2}\right\}, \tag{3.72}$$

and hence, after making the change of variables (3.71), we find the following distribution function for the excess area:

$$F(S) = 1 - \exp\left\{-\frac{1}{2}\left(\frac{3}{2}\frac{b^2}{\sigma^3}\sqrt{R_2}S\right)^{2/3}\right\}. \tag{3.73}$$

The probability density corresponding to (3.73) equals

$$w(S) = \frac{1}{3}\left(\frac{3}{2}\frac{b^2}{\sigma^3}\sqrt{R_2}\right)^{2/3} S^{-1/3} \exp\left\{-\frac{1}{2}\left(\frac{3}{2}\frac{b^2}{\sigma^3}\sqrt{R_2}S\right)^{2/3}\right\}, \tag{3.74}$$

and is shown graphically in Figure 7, in terms of the constant

$$\varrho = \frac{3}{2}\frac{b^2}{\sigma^3}\sqrt{R_2}.$$

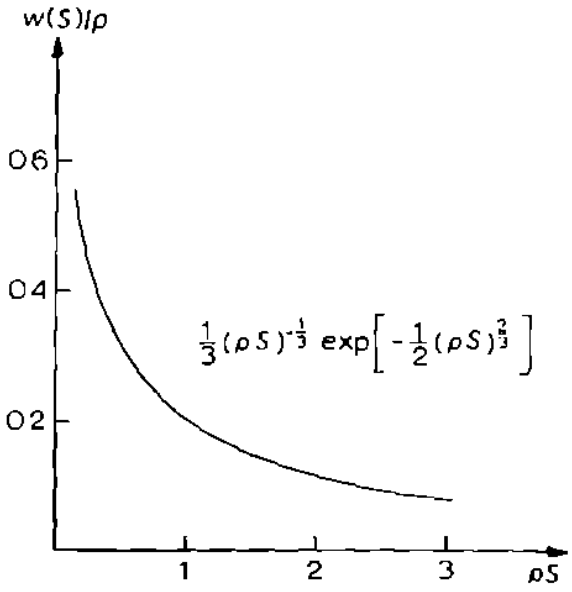

Fig. 7. The probability density of the excess area of the peaks.

To find the fraction of skipped peaks, which do not activate the relay, we need only substitute the minimum area S_m into (3.73). Then, subtracting the result from the average total number of peaks and recalling (1.9), we obtain the number of peaks which actually cause the relay to operate:

$$n(S_m) = n_0 \exp\left\{-\frac{1}{2}\left(\frac{3}{2}\frac{b^2}{\sigma^3}\sqrt{R_2}\,S_m\right)^{2/3}\right\} = \frac{\sqrt{R_2}}{2\pi}\exp\left\{-\frac{b^2}{2\sigma^2}\left[1+\left(\frac{3}{2}\frac{\sqrt{R_2}}{b}S_m\right)^{2/3}\right]\right\}. \tag{3.75}$$

It is apparent from (3.75) and (3.28) that the insensitivity of a relay to low-energy or short-duration peaks is equivalent to an increase in the operating level by a quantity which depends on the properties of the noise, more exactly on

$$R_2 = -\frac{d^2R(0)}{d\tau^2}.$$

4. Effect of Pulse Signals on a Relay in the Presence of Noise. The Dead Time.

Electronic relays are used to register and count pulses of known shape $s(t)$, which together constitute a useful signal $s_1(t)$, usually contaminated by the presence of additive noise $\xi(t)$. This is the case when relays are used in digital computers, in radar equipment with digital display, in devices for measuring the intensity of radioactive radiation, in apparatus for the precise measurement of short time intervals, in various systems for coding, pulse synchronization, automatic control, etc.

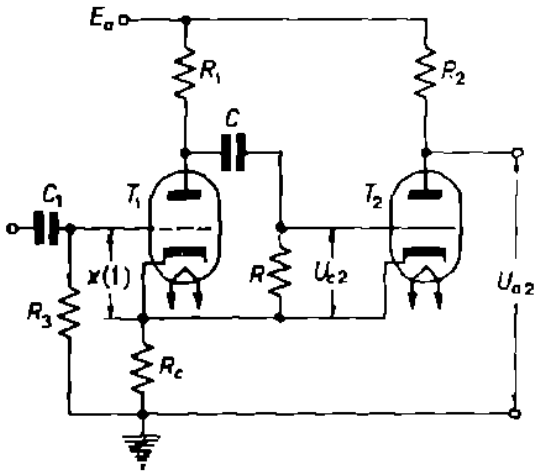

Fig. 8. Circuit diagram of an electronic relay.

As an example, consider Figure 8, which shows the circuit diagram of an electronic relay with a single equilibrium state. Let the input signal be the voltage $x(t)=s_1(t)+\xi(t)$ between the grid and cathode of the first tube T_1. In the absence of an external voltage, the relay is in a state of stable equilibrium, with the tube T_1 drawing no current because of the bias voltage produced in the resistance R_c by the flow of anode current I_{a2} through the tube T_2. As soon as the external voltage acting on the grid of the tube T_1 reaches a certain critical value b, the relay "trips," i.e., the tube T_1 begins to draw current and the tube T_2 ceases to draw current. Every time it operates, the relay

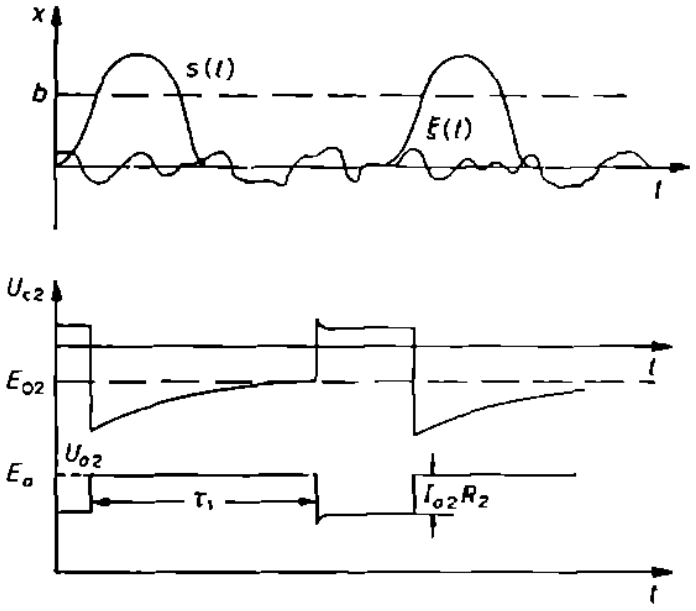

Fig. 9. Behavior of voltages in the relay of Fig. 8.

produces a voltage pulse across the resistance R; this voltage U_{c2} subsequently discharges through the condenser C and the tube T_1 until it reaches the value E_{02} at which T_2 begins to draw current. When this happens, the relay returns to its original state. The evolution in time of the sequence of events just described is shown in Figure 9.

The parameters of the useful signal, in particular the height of the pulses of which it consists, are chosen in such a way that each separate pulse activates the relay. Therefore, in the absence of noise, the relay operates a number of times exactly equal to the number of pulses appearing at its input. The presence of noise produces a twofold effect: In the first place, the relay sometimes operates when no pulse is present, i.e., a peak of the contaminating noise may be misinterpreted as a signal pulse; this effect will be called a "false alarm." Secondly, a signal pulse may be "skipped" altogether; this happens when the sum of the signal and noise turns out to be too small to activate the relay.

The average total number of false alarms due to noise peaks in the absence of signal pulses is the quantity n_0 already found in Chap. 1 [cf. (1.5) and (1.9)]. If the relay does not respond to all the peaks, but only to those of sufficiently long duration ($\tau > \tau_m$) or of sufficiently large excess area ($S > S_m$), then we can use formula (3.28) or (3.75),

provided the noise is smoothly varying and the operating level is high enough. However, in deriving these formulas, we have not yet taken account of a characteristic feature of relay operation, i.e., the existence of a "dead time." This is just the period of time during which the relay is still undergoing the transient produced by its response to an earlier pulse and hence is not yet ready to respond to a new pulse. Any new pulse arriving during this period will be skipped. The simplest way of dealing with this phenomenon is to assume that there exists a time constant τ_1 characterizing the minimum interval that must elapse between initial points of peaks before the relay can respond to a new pulse. In other words, a pulse satisfying all the other requirements will not cause the relay to operate if the interval between its initial point and the initial point of a previous pulse which activated the relay is less than τ_1. Of course, more complicated behavior is possible, and the requirements for reactivating the relay (e.g., the minimum excess area) may depend continuously on the time since the last activation of the relay.

The effect just described compels us to take account of the distribution of the lengths of intervals between peaks. The complicated problem of calculating this distribution can be simplified in the case of a high operating level, where the correlation between the different peaks becomes weak. Then the only peaks which are appreciably correlated are separated by short time intervals $\Delta t \lesssim \tau_{st}$ (st for "short time"). Here τ_{st} is a kind of correlation time for peaks, and describes the region of enhanced conditional probability of a new peak occurring, given that a previous peak has occurred. This correlation time decreases as b is increased, and for $b - \langle x \rangle \gg \sigma(x)$ becomes much less than the correlation time τ_{cor} of the original process $x(t)$. For large time intervals $\Delta t \gg \tau_{st}$, we can neglect the correlation between peaks, regarding the appearance of different peaks as independent events. This absence of correlation corresponds to a Poisson distribution

$$w(T_{int}) = n_0 e^{-n_0 T_{int}} \tag{3.76}$$

for the intervals T_{int} between adjacent peaks (because of the short duration of the peaks, T_{int} is approximately equal to the interval between initial points of adjacent peaks). For $T_{int} \lesssim \tau_{st}$, the probability density $w(T_{int})$ has somewhat larger values than given by (3.76). For a

sufficiently large dead time $\tau_1 \gg \tau_{sl}$, this effect can be neglected and the different peaks can be regarded as uncorrelated.

Making this assumption, we now calculate the number of false alarms, with due regard for the dead time. Let n_1 be the mean number of times the relay operates (in the absence of signal pulses), and let n be the same quantity in the absence of any dead time effect. Then during a time T, the relay operates $n_1 T$ times, and each operation of the relay gives rise to a dead time τ_1. To take full account of the effect of the dead time, we need only note that the relay is insensitive for a total time equal to $n_1 T\tau_1$, and hence capable of responding only for a time equal to $T(1-n_1\tau_1)$. During this time, $nT(1-n_1T)$ peaks occur which are capable of causing the relay to operate, and these peaks actually cause the relay to operate $n_1 T$ times. It follows that

$$n_1 T = nT(1 - n_1\tau_1),$$

or

$$n_1 = \frac{n}{1 + n\tau_1}. \tag{3.77}$$

If $n\tau_1 \ll 1$, the effect of the dead time is slight, and we have

$$n_1 = n - n^2\tau_1, \tag{3.78}$$

which is close to n. In the opposite case, where $n\tau_1 \gg 1$, formula (3.77) gives $n_1 \approx 1/\tau_1$. This is natural, since the interval between successive operations of the relay cannot be less than τ_1.

The presence of signal pulses increases the total time during which the relay is insensitive, and this decreases the number of false alarms compared to (3.77). In fact, if n_2 is the actual rate of false alarms in the presence of signal pulses, and if the latter occur at the rate n_p, then the total dead time is now $n_2 T\tau_1 + n_p T\tau_1$. Subtracting this expression from the total time and multiplying by n, we again find the actual number of false alarms:

$$n_2 T = n(T - n_2 T\tau_1 - n_p T\tau_1). \tag{3.79}$$

It follows that

$$n_2 = \frac{1 - n_p\tau_1}{1 + n\tau_1}. \tag{3.80}$$

This formula is valid for any relation between $n_p\tau_1$, $n\tau_1$ and unity, provided, of course, that $(n_p+n)\tau_1 < 1$. In most practical cases, there are comparatively few false alarms, i.e., $n_2 \ll n_p$, and then we can use the condition $n\tau_1 \ll 1$ to deduce from (3.80) the approximate formula

$$n_2 = (1 - n_p\tau_1 - n\tau_1)n + O(\tau_1^2). \tag{3.81}$$

As already noted, the quantity n figuring in (3.77)–(3.81) can be found by using the formulas derived earlier.

Having considered the case of false alarms, we now turn to the other effect of noise on relays, i.e., the skipping of signal pulses. As a rule, pulses are skipped rather rarely, and in fact, the height and length of the signal pulses are generally chosen to be quite large, in order to keep down the number of skipped pulses. Here we give only a qualitative formula, suitable for evaluating the rate at which pulses are skipped, since there is hardly any practical need for a more exact formula.

Suppose the pulse has a rectangular shape and begins at time zero:

$$s(t) = \begin{cases} B & \text{for } 0 \leqslant t \leqslant \tau_p, \\ 0 & \text{otherwise}. \end{cases} \tag{3.82}$$

The voltage at the input of the relay is the sum of $s(t)$ and the contaminating noise $\xi(t)$ [see Figure 10]:

$$x(t) = s(t) + \xi(t). \tag{3.83}$$

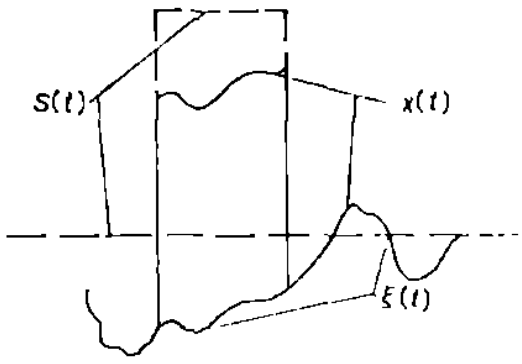

Fig. 10. An input signal equal to the sum of a rectangular pulse and contaminating noise.

Suppose the relay has operating level $b<B$. (We ignore niceties connected with minimum peak duration or minimum excess area.) Then a signal pulse will be skipped if and only if the curve $x(t)$ lies below the level b during the whole pulse, i.e.,

$$\xi(t) < b - B \tag{3.84}$$

for $0 \leqslant t \leqslant T_p$. If we were only interested in the probability P of this inequality holding at one instant of time, then obviously

$$P = \int_{-\infty}^{b-B} w(\xi)\, d\xi\,. \tag{3.85}$$

However, we need to know the probability of (3.84) holding at any time during the pulse, and this is a more complicated problem. By partitioning the interval $[0, T]$, the probability of (3.84) holding in the interval $[0, T]$ is given formally by the integral

$$P = \int_{-\infty}^{b-B} \cdots \int_{-\infty}^{b-B} w\left[\xi(t_1), \ldots, \xi(t_N)\right] d\xi(t_1) \ldots d\xi(t_N) \qquad (0 \leqslant t_1 < \ldots < t_N \leqslant T), \tag{3.86}$$

which it is difficult to evaluate. A simple, but not very precise formula, which is suitable for practical purposes, can be obtained by replacing $\xi(t)$ by a random step function which changes its values by jumps after intervals of the order of τ_{cor} (see Figure 11), where we assume that the values of the function $\xi(t)$ are statistically independent even when only one jump apart.

With this assumption, the probability (3.84) equals some power of the integral (3.85). For example, if $\tau_p = 0$, (3.86) coincides with (3.85)

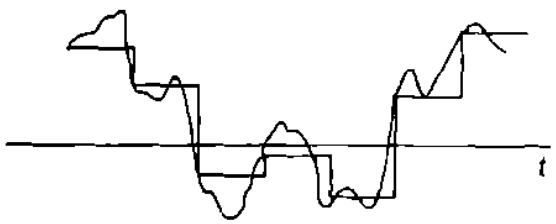

Fig. 11. Approximation of continuous noise by a random step function.

and hence equals the first power of (3.85), while if $\tau_p = \tau_{cor}$ (3.86) equals the second power of (3.85), and so on, with the exponent undergoing a jump whenever τ_p is an integral multiple of τ_{cor}. Thus, for integral values of τ_p/τ_{cor}, we have

$$P = \left[\int_{-\infty}^{b-B} w(\xi)\,d\xi\right]^{1+(\tau_p/\tau_{cor})} = [F_\xi(b-B)]^{1+(\tau_p/\tau_{cor})}. \tag{3.87}$$

and we continue to use the same formula if τ_p/τ_{cor} is not an integer. Assuming that the effect of the additive noise on each pulse is statistically independent of its effect on the other pulses, we conclude that (3.86) and (3.87) give the fraction of skipped pulses. If the average total number of skipped pulses per unit time equals n_p, then the number of skipped pulses, which do not "trip" the relay, equals

$$n_{skip} = n_p[F_\xi(b-B)]^{1+(\tau_p/\tau_{cor})}, \tag{3.88}$$

while the number of times the pulses cause the relay to operate obviously equals

$$n_{op} = n_p - n_{skip}. \tag{3.89}$$

In the case of Gaussian noise with mean zero and variance σ^2, we have

$$w(\xi) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\xi^2/2\sigma^2},$$

and hence

$$F_\xi(b-B) = \frac{1}{2}\operatorname{erfc}\left(\frac{B-b}{\sqrt{2}\sigma}\right).$$

It follows that[3]

$$n_{skip} = n_p\left[\frac{1}{2}\operatorname{erfc}\frac{B-b}{\sqrt{2}\sigma}\right]^{1+(\tau_p/\tau_{cor})} \tag{3.90}$$

Formula (3.90) shows that the number of skipped pulses grows as $B-b$ is increased or as the pulse duration is increased compared to the

[3] If the correlation coefficient $R(\tau) = e^{-\gamma|\tau|}$, then $\tau_{cor} = 1/\gamma$, while if $R(\tau) = e^{-\alpha\tau^2}$, then $\tau_{cor} = \sqrt{\pi/4\alpha} = \sqrt{1/\alpha}$ [cf. (I.2.7)].

correlation time. In the case $B-b \gg \sigma$, we can use the asymptotic formula

$$\operatorname{erfc} x = \frac{e^{-x^2}}{\sqrt{\pi} x},$$

obtaining

$$n_{skip} = n_p \left[\frac{\sigma}{\sqrt{2\pi}(B-b)} \right]^{1+(\tau_p/\tau_{cor})} \exp\left\{ -\frac{(B-b)^2}{2\sigma^2} \frac{\tau_p + \tau_{cor}}{\tau_{cor}} \right\}. \quad (3.91)$$

5. Jitter of Relay Operating Time Due to the Presence of Noise

There are many applications in which an electronic relay is used to produce an output pulse whenever a signal pulse $s(t)$ is applied to its input. Besides false alarms and skipped signal pulses, the presence of additive random noise produces still another effect. We refer to "jitter" of the relay operating time, i.e., instability of the moment at which the relay responds to a signal pulse at its input. In the absence of noise, it can be assumed that the relay always begins to operate at the same point of the waveform $s(t)$, so that the time t_0 at which the relay operates can be used to locate the exact position of the input pulse along the time axis. The presence of noise leads to a random time shift of the output pulse relative to the input pulse, and hence t_0 can no longer be used to deduce the exact time of arrival of the input pulse. Of course, noise in the relay itself (e.g., power supply instability) leads to the same effect, since such noise causes instability in the operating level b. However, by ascribing this internal noise to the random input signal, we can regard the operating level b as constant.

Thus, in any event, suppose a signal of the form

$$x(t) = s_1(t) + \xi(t)$$

is applied to the input of an ideal relay, where $\xi(t)$ represents random noise, and, as before, $s_1(t)$ is a sum of pulses of standard form:

$$s_1(t) = \sum_j s(t - t_j).$$

Consider a single pulse $s(t)$, choosing the "centering time" $t_j = 0$. To

calculate the jitter in the relay operating time, we argue as follows: The operating time t_{op} in the presence of noise is the moment at which the curve $x(t)$ crosses the level b, and hence satisfies the equation

$$s(t_{op}) + \xi(t_{op}) = b. \tag{3.92}$$

If $\xi(t) \equiv 0$, equation (3.92) reduces to

$$s(t_0) = b, \tag{3.93}$$

where t_0 is the operating time in the absence of noise. Subtracting (3.93) from (3.92), we obtain

$$u(t_{op}) + \xi(t_{op}) = 0, \tag{3.94}$$

where

$$u(t_{op}) = s(t_{op}) - s(t_0).$$

The signal pulse and the noise are both smoothly varying functions, since they arrive at the relay input only after going through some system with finite bandwidth. Therefore the rate of change of $s(t)$ and $\xi(t)$ is characterized by some time constant τ_{sm}, which means that the terms of the Taylor series of $s(t)$ and $\xi(t)$ fall off like powers of the ratio $\Delta t/\tau_{sm}$.

The next step is to solve (3.94) for the operating time t_{op} by using the method of successive approximations, under the assumption that the noise $\xi(t)$ is weak. To make the method of solution more transparent, we equip the random function $\xi(t)$ figuring in (3.94) with a small parameter ε, writing (3.94) in the form

$$u(t_{op}) + \varepsilon\xi(t_{op}) = 0. \tag{3.95}$$

Then at the end of the calculation, we set $\varepsilon = 1$, bearing in mind that the method of solution is justified by the smallness of $\xi(t)$.[4]

Thus, writing the solution of (3.95) in the form

$$t_{op} = t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \ldots, \tag{3.96}$$

and then substituting (3.96) into (3.95), we have

$$u(t_0 + \varepsilon t_1 + \varepsilon t^2 + \ldots) + \varepsilon\xi(t_0 + \varepsilon t_1 + \ldots) = 0.$$

[4] This statement will be made more precise later.

After expanding the functions u and ξ in Taylor series, this becomes

$$u'(t_0)(\varepsilon t_1 + \varepsilon^2 t_2 + \ldots) + \tfrac{1}{2}u''(t_0)(\varepsilon t_1 + \varepsilon^2 t_2 + \ldots)^2 + \ldots + \varepsilon\xi(t_0) + \varepsilon\xi'(t_0)(\varepsilon t_1 + \ldots) + \ldots = 0$$

[recall that $u(t_0)=0$]. Equating the coefficients of identical powers of the parameter ε, we obtain the consecutive approximations

$$\begin{aligned} u't_1 + \xi &= 0, \\ u't_2 + \tfrac{1}{2}u''t_1^2 + \xi' t_1 &= 0, \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{3.97}$$

where $\xi = \xi(t_0)$ and $u' = u'(t_0)$, or equivalently,

$$t_1 = -\frac{\xi}{u'}, \tag{3.98}$$

$$t_2 = -\frac{\xi'}{u'}t_1 - \frac{1}{2}\frac{u''}{u'}t_1^2 = \frac{\xi\xi'}{u'^2} - \frac{1}{2}\frac{u''\xi^2}{u'^3}. \tag{3.99}$$

Finally, recalling (3.96), we write the solution of (3.95) in the form

$$t_{op} - t_0 = -\varepsilon\frac{\xi}{u'} + \varepsilon^2\left(\frac{\xi\xi'}{u'^2} - \frac{u''\xi^2}{2u'^3}\right) + O(\varepsilon^3),$$

or

$$t_{op} - t_0 = -\frac{\xi}{s'} + \frac{\xi\xi'}{s'^2} - \frac{s''\xi^2}{2s'^3}, \tag{3.100}$$

where we set $\varepsilon = 1$ and neglect terms of order 3 or higher.

It follows from (3.100) that the operating time is shifted by an amount depending on the derivatives of the functions $s(t)$ and $\xi(t)$ at the time t_0. The first term in the right-hand side of (3.100) stems from the first approximation, and the other two terms give the second-order correction. An analysis of (3.100), and also of the higher-order approximations, leads to the conclusion that the successive approximations converge like powers of the ratio ξ'/s', which equals $\sigma(\xi)/s'\tau_{sm}$ in order of magnitude. Here we use the fact that each differentiation of s and ξ is in order of magnitude equivalent to dividing by the "smoothing time" τ_{sm}. Thus the method of successive approximations will be very effective if

$$\frac{\sigma(\xi)}{s'\tau_{sm}} \ll 1.$$

We conclude this section by using the above results to calculate the statistical characteristics of the time shift (3.100) for the case of Gaussian noise with zero mean and correlation function $\langle \xi \xi_\tau \rangle = \sigma^2 R(\tau)$. Averaging (3.100), we obtain a nonzero mean shift

$$\langle t_{op} - t_0 \rangle = -\frac{s''}{2s'^3}\sigma^2, \tag{3.101}$$

proportional to the second derivative of the pulse waveform. The quantity s'' is usually negative, since the operating level is usually closer to the top than to the bottom of the pulse $s(t)$. Therefore $\langle t_{op} - t_0 \rangle > 0$, i.e., on the average the noise produces a small time lag in the operating time of the relay. Moreover, taking the square of (3.100) and averaging the result, we find that

$$\mathbf{D}(t_{op} - t_0) = \frac{\sigma^2}{s'^2} + \frac{\sigma^4 R_2}{s'^4} + \frac{3\sigma^4 s''^2}{4s'^6}, \tag{3.102}$$

where

$$R_2 = -\frac{d^2 R(0)}{d\tau^2}.$$

In deriving (3.102), we have used the statistical independence of ξ and ξ', and the first of the formulas (I.8.31). It is apparent that the jitter of the operating time is largely determined by σ, the standard deviation of the noise, and s', the steepness of the signal wavefront. The second-order corrections, which depend on s'' and R_2 (and hence on the rate of change of the contaminating noise), lead to further instability of the relay operating time. The convergence of the series for $\mathbf{D}(t_{op} - t_0)$, whose first few terms are given by (3.202), is characterized by the parameter $\sigma^2 R_2 / s'^2$.

Equation (3.100) can also be used to make an approximate calculation of the probability density of the time shift $t_{op} - t_0$. For convenience, we first introduce the dimensionless quantity

$$\tau = s' \frac{t_{op} - t_0}{\sigma} = -\frac{\xi}{\sigma} + \frac{\xi \xi'}{\sigma s'} - \frac{s'' \xi^2}{2\sigma s'^2}, \tag{3.103}$$

instead of $t_{op} - t_0$. In the first approximation, where $\tau = -\xi/\sigma$, we have

a normal distribution

$$w(\tau) = \frac{1}{\sqrt{2\pi}} e^{-\tau^2/2}. \tag{3.104}$$

The other two terms in (3.103) modify the distribution (3.104). The larger of these corrections to (3.104) is due to the term

$$-\frac{s''\xi^2}{2\sigma s'^2},$$

In fact, substituting (3.103) into the characteristic function $\langle e^{iv\tau}\rangle$ and expanding the result in a series with respect to

$$\frac{\xi\xi'}{\sigma s'} - \frac{s''\xi^2}{2\sigma s'^2},$$

we obtain

$$\langle e^{iv\tau}\rangle = \left\langle e^{-iv\zeta/\sigma}\left(1 + iv\frac{\xi\xi'}{\sigma s'} - iv\frac{\sigma s''}{2s'^2}\frac{\xi^2}{\sigma^2} + \dots\right)\right\rangle. \tag{3.105}$$

When the average is carried out, the term in $\xi\xi'$ disappears, since ξ' is independent of ξ and $\langle\xi'\rangle = 0$. After dropping this term and the higher-order terms indicated by $+\dots$, (3.105) becomes

$$\begin{aligned}\langle e^{iv\tau}\rangle &= \langle e^{-iv\zeta/\sigma}\rangle + iv\frac{\sigma s''}{2s'^2}\frac{\partial^2}{\partial v^2}\langle e^{i-v\zeta/\sigma}\rangle \\ &= \left[1 + \frac{\sigma s''}{2s'^2} iv(v^2 - 1)\right] e^{-v^2/2}.\end{aligned} \tag{3.106}$$

Taking the Fourier transform of (3.106), we obtain the probability density

$$\begin{aligned}w(\tau) &= \left[1 + \frac{\sigma s''}{2s'^2}\left(\frac{\partial^3}{\partial\tau^3} + \frac{\partial}{\partial\tau}\right)\right]\frac{1}{\sqrt{2\pi}} e^{-\tau^2/2} \\ &= \left[1 + \frac{\sigma s''}{2s'^2}(2\tau - \tau^3)\right]\frac{1}{\sqrt{2\pi}} e^{-\tau^2/2}.\end{aligned} \tag{3.107}$$

If necessary, the same method can be used to derive further corrections to the distribution of τ. It should be noted that the corrected distribution (3.107) is no longer symmetric.

PART 2

Nonlinear Self-Excited Oscillations in the Presence of Noise

CHAPTER 4

Basic Equations Describing the Operation of an Oscillator in the Presence of Noise

1. Preliminary Remarks

Sine wave generators are used for a variety of purposes in contemporary engineering practice, e.g., for producing the carrier frequency in a radio transmitter, for changing the carrier frequency of the input signal in a superheterodyne receiver, for producing standard signals for applications to radio astronomy, radar, navigation, etc. The sine wave generators employed in radio engineering vary greatly as to the generated frequency, the power, and the physical process used to produce the self-excited oscillations. However, a common feature of all such oscillators is that they use *positive feedback*, which must be strong enough to amplify and sustain the initial perturbations. Another characteristic feature of most oscillators is the presence of a frequency-selective element (a tank circuit), with a resonant frequency near that of the generated frequency. Without such an element, the self-excited oscillations, instead of being sinusoidal, would have a more complicated form, like the relaxation oscillations of a multivibrator or blocking oscillator.

Various physical phenomena can be used to produce self-excited oscillations, and correspondingly, the basic equations describing the operation of the oscillator can be quite diverse. In this book, however, we shall confine ourselves exclusively to the case of oscillators

whose operation is described by a second-order differential equation. Such an equation can always be reduced to the form

$$\ddot{x} = f_0(x, \dot{x}, t), \tag{4.1}$$

where $x(t)$ is some coordinate, e.g., the voltage or current in a given circuit element, while $f_0(x, \dot{x}, t)$ is some function whose form is determined by the specific nature of the oscillator, i.e., its circuit diagram.[1] Examples of such functions will be given below.

As already suggested, there are generators which cannot be described by a second-order differential equation. For example, if the frequency-selective element consists of several resonant circuits, the basic equation may be of order higher than 2. In other instances, the operation of the oscillator may be described by an integro-differential equation, as in the case of oscillators which use semiconductor devices. Finally, the generation of ultra-high frequency oscillations in distributed-parameter systems is described by partial differential equations which reduce to ordinary differential equations only in certain approximations. These more complicated situations will not be discussed here.

The oscillators considered in this book all belong to the category of [illegible]enerators, i.e., oscillators whose output is close to being sinusoidal. There are two ways in which the output of such a device may fail to be sinusoidal. In the first place, there are high-frequency deviations. In this case, the output signal is close to a sine wave of a certain amplitude and phase during one or several periods, but takes values both above and below the sine wave, with the deviations corresponding to a frequency higher than the basic frequency of the oscillator. These high-frequency deviations are small in absolute value, or else can be made small after preliminary smoothing. Secondly, a signal can differ from a sine wave because of slowly varying but constantly accumulating deviations of amplitude and phase, which are small for one or several periods, but, after a large number of periods, the amplitude and phase change completely, and the signal goes into a new sine wave, different from the original one. This second kind of departure

[1] As usual, the overdot denotes differentiation with respect to time.

from pure sinusoidal behavior is often more important than the first, since in the circuits usually encountered in engineering practice, the high-frequency deviations are as a rule filtered out, and hence are of interest only insofar as they affect the slow changes of amplitude and phase.

Now let ω_0 be an approximate angular frequency, determined from a few arbitrarily chosen "periods" of the oscillations. If the generated signal $x(t)$ were purely sinusoidal, it would satisfy the equation

$$\ddot{x} = -\omega_0^2 x. \tag{4.2}$$

Clearly, if $x(t)$ is not a pure sine wave but if the deviations of the kinds just discussed are small, then the right-hand sides of equations (4.1) and (4.2) differ only slightly. To clarify the situation, we introduce a small parameter ε and write

$$f_0(x, \dot{x}, t) = -\omega_0^2 x + \varepsilon f(x, \dot{x}, t).$$

Then the equation (4.1) satisfied by the oscillations becomes

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}, t). \tag{4.3}$$

All that has just been said is also valid in the case where external or internal excitations are applied to the circuit elements. This is taken into account by having f depend explicitly on t. Such excitations may be regular, as in the case where a voltage pulse is applied to the anode of a vacuum-tube oscillator, thereby generating a high-frequency signal pulse. Another example is the case where a periodic external excitation is used to stabilize (synchronize) the oscillator frequency. On the other hand, there may also be "irregular" excitations, due to instability of the circuit parameters, power supply instability, internal shot noise and thermal noise, as well as external random noise, and in all such cases, the time-dependent excitations must be described statistically. Then the generated signal will also be a random function of time, whose statistical characteristics have to be determined.

From a mathematical standpoint, in the presence of random excitations, equation (4.3) represents a nonlinear equation containing a random function. However, because of the smallness of the parameter ε, equation (4.3) is only slightly nonlinear, and essential use of this fact is made in solving the problem. Exploiting the smallness of ε, we can

use asymptotic methods, which allow us to obtain results in the form of expansions with respect to ε to any order.

2. An Example of a Self-Excited System. The Vacuum-Tube Oscillator

As an example of a self-excited system, consider the vacuum-tube oscillator with the circuit diagram shown in Figure 12. Here the anode circuit contains an LC-circuit, and the capacitances C_1 and C_2 are chosen to have negligibly small a–c resistance, while, on the contrary, the resistance R_1 is quite large. Therefore Figure 12 is equivalent to the simplified circuit diagram shown in Figure 13, where the constant anode voltage and grid bias are not explicitly indicated.

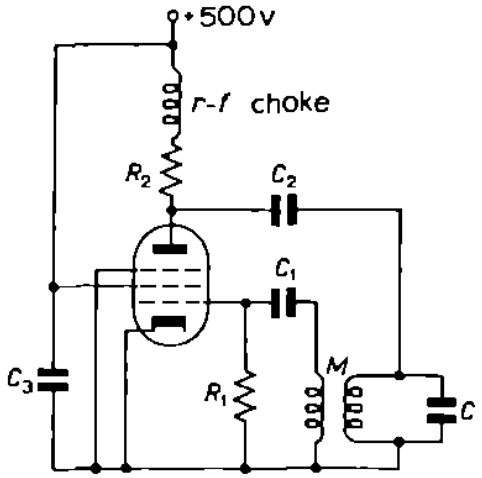

Fig. 12. Circuit diagram of a vacuum-tube oscillator.

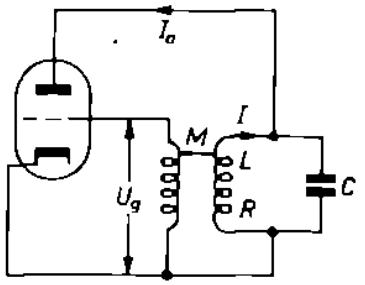

Fig. 13. Simplified circuit diagram corresponding to Fig. 12.

Denoting the current in the inductive branch of the LC-circuit by I, and equating the voltages across the inductive and capacitive branches of the circuit, we obtain the equation

$$L\frac{dI}{dt} + RI = \frac{1}{C}\int (I_a - I)\,dt$$

or

$$\ddot{I} + \omega_0^2 I = -\omega_0^2 RC\dot{I} + \omega_0^2 I_a, \tag{4.4}$$

where

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

Next we express the anode current I_a in terms of the current I. For a given grid bias and anode voltage, the anode current is a known function

$$I_a = F(U_g, U_a) \tag{4.5}$$

of the a–c components of the grid and anode voltages. By introducing the coefficient D (the reciprocal of the amplification factor), this dependence is often represented as a function of one variable, i.e.,

$$I_a = F(U), \quad \text{where} \quad U = U_g + DU_a. \tag{4.6}$$

If we neglect the grid current, the a–c component of the grid voltage will be determined exclusively by the e.m.f. introduced by the presence of the mutual inductance M:

$$U_g = M\dot{I}. \tag{4.7}$$

The a–c component of the anode voltage equals the voltage across the LC-circuit, taken with the opposite sign. Using this fact, and substituting (4.6) and (4.7) into (4.4), we finally obtain the following basic equation describing the operation of the oscillator:

$$\ddot{I} + \omega_0^2 I = -\omega_0^2 RC\dot{I} + \omega_0^2 F(M\dot{I} - DL\dot{I} - DRI). \tag{4.8}$$

Equation (4.8) is of the form (4.3), but the right-hand side does not explicitly depend on time. Time dependence will appear if we take account of internal noise or introduce external excitations. For example, because of fluctuations due to shot noise, the instantaneous

value of the anode current I_a differs by a random amount from the average anode current $\langle I_a \rangle$ corresponding to the grid and anode voltages at the given instant. In fact, equations (4.6) and (4.7) hold only for the average anode current

$$\langle I_a \rangle = F(U_g + DU_a), \tag{4.9}$$

and the actual plate current

$$I_a = F(U_g + DU_a) + I_f(t) \tag{4.10}$$

has an additional fluctuational component $I_f(t)$, whose spectral density is given by formula (I.6.41). Substituting (4.10) into (4.4), we obtain the equation

$$\ddot{I} + \omega_0^2 I = \omega_0^2 [-RC\dot{I} + F(M\dot{I} - DL\dot{I} - DRI) + I_f(t)], \tag{4.11}$$

which differs from (4.3) by the presence of an extra fluctuational term in the right-hand side. If we set $D=0$, thereby neglecting the reaction of the anode load, (4.11) simplifies to

$$\ddot{I} + \omega_0^2 I = \omega_0^2 [-RC\dot{I} + F(M\dot{I}) + I_f(t)]. \tag{4.12}$$

The effect of the anode load can be largely taken into account by replacing M by $M' = M - DL$ in (4.12).

Next we consider external excitations, which can be introduced in various ways. In Figure 14 we show an extra a–c anode voltage $u_a(t)$,

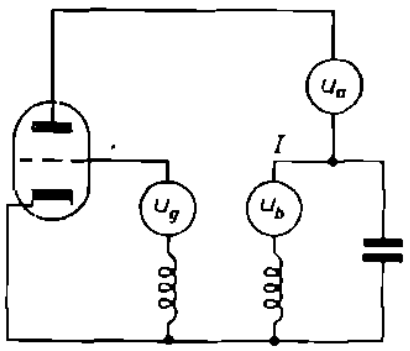

Fig. 14. Application of external excitations to an oscillator.

an extra grid voltage $u_g(t)$ and an extra voltage $u_b(t)$ across the inductive branch of the LC-circuit, where $u_b(t)$ can be introduced by coupling L with another coil. To avoid repetition of calculations which closely resemble one another, we assume that all three external voltages are present simultaneously; afterwards any voltage which is not actually present can be set equal to zero in the final equation. Equating the voltages across the two branches of the LC-circuit, we find that

$$L\dot{I} + RI - u_b(t) = \frac{1}{C}\int (I_a - I)\, dt\,.$$

Since the grid voltage is now $M\dot{I} + u_g$, the equation describing the operation of the oscillator (in the absence of shot noise) becomes

$$\ddot{I} + \omega_0^2 I = \omega_0^2 [-RC\dot{I} + F(M\dot{I} - DL\dot{I} - DRI + u_g + Du_b + Du_a) + C\dot{u}_b]\,. \tag{4.13}$$

For a tank circuit with high Q, we can neglect the term RI compared to $L\dot{I} \sim L\omega_0 I$. Then, setting

$$u = u_g + Du_a + Du_b\,, \tag{4.14}$$

we write (4.13) in the form

$$\ddot{I} + \omega_0^2 I = \omega_0^2 \{-RC\dot{I} + F[M'\dot{I} + u(t)] + C\dot{u}_b\}\,, \tag{4.15}$$

where $M' = M - DL$ can be replaced by M if the reaction of the anode load is negligible. The external excitations may be regular or fluctuational in nature. For example, if

$$u_g = u_a = 0\,, \quad u_b = U_0 \cos \omega_s t\,, \tag{4.16}$$

then, taking account of shot noise and neglecting the reaction of the anode load ($D = 0$), we have

$$\ddot{I} + \omega_0^2 I = \omega_0^2 [-RC\dot{I} + F(M\dot{I}) - \omega_s C U_0 \sin \omega_s t + I_f(t)]\,. \tag{4.17}$$

To make (4.11), (4.15) and (4.17) resemble (4.3) completely, we need only introduce a small parameter ε. Before doing this, we have to get rid of the d–c component $I_=$ of the anode current, which can sometimes be quite large. This can be achieved either by subtracting out $I_=$ or by differentiating both sides of the relevant equation with respect to time.

The latter is the appropriate procedure when $I_=$ varies appreciably over several periods even though it is substantially constant during a single period, and after carrying out the differentiation, it is convenient to write the resulting equation in terms of the grid voltage $U_g = MI$. However, if $I_=$ does not change appreciably even over a large number of periods, it is best to write our equations in terms of $I - I_=$, the a–c component of the current in the inductive branch of the LC-circuit, or, what amounts to the same thing, in terms of the quantity

$$x = \omega_0 M (I - I_=) \tag{4.18}$$

with the dimensions of voltage. For pure sinusoidal oscillations, this voltage has the same amplitude as the a–c voltage on the grid of the tube. Let $F_0(U)$ be the tube characteristic after the d–c component of the anode current has been subtracted out, i.e.,

$$\langle I_a \rangle - I_= = F_0(U), \quad \text{where} \quad U = U_g + DU_a. \tag{4.19}$$

Then, for example, using (4.18) and (4.19), we can write (4.12) in the form

$$\ddot{x} + \omega_0^2 x = \omega_0^2 [\omega_0 M F_0(\dot{x}/\omega_0) - RC\dot{x} + \omega_0 M I_f(t)]. \tag{4.20}$$

In exactly the same way, equation (4.15) corresponding to the presence of an external excitation $u_b(t)$ [with $u_g = 0$, $D = 0$] becomes

$$\ddot{x} + \omega_0^2 x = \omega_0^2 [\omega_0 M F_0(\dot{x}/\omega_0) - RC\dot{x} + \omega_0 MC\dot{u}_b(t)]. \tag{4.21}$$

It is often convenient to approximate the tube characteristic by a cubical parabola

$$F(U) = \ _= + F_0(U) = I_= + SU + \frac{\beta}{2} U^2 - \frac{\gamma}{3} U^3, \tag{4.22}$$

where S is the "steepness" (transconductance) and γ is a parameter sometimes written in the form S/V^3. If the characteristic is symmetric with respect to the operating point, we can use the simpler expression

$$F(U) = SU - \frac{\gamma}{3} U^3. \tag{4.23}$$

Which of the approximations (4.22) and (4.23) should be used is only a detail affecting the final calculations. The simpler expression (4.23) can

sometimes be used even when the characteristic (4.22) is asymmetric, since it turns out that the term $\beta u^2/2$ has no effect in the first approximation [see the discussion of equations (6.6) and (6.7)]. Using (4.23), we can write (4.20) and (4.21) in the form

$$\ddot{x} + \omega_0^2 x = \varepsilon\omega_0\left(1 - \frac{4\dot{x}^2}{3\omega_0^2 A_0^2}\right)\dot{x} + \omega_0^2\varepsilon\xi(t). \tag{4.24}$$

Here we have introduced the following notation: $\varepsilon = \omega_0(MS - RC)$ is a small parameter, the quantity

$$A_0 = 2\left(\frac{MS - RC}{M\gamma}\right)^{1/2} = 2\left(\frac{\varepsilon}{\omega_0 M\gamma}\right)^{1/2} \tag{4.25}$$

as will be apparent later, can be interpreted as the amplitude of the oscillations of grid voltage [cf. (4.41)], and $\xi(t)$ is a random function equal to

$$\xi(t) = \frac{\omega_0 M I_f}{\varepsilon} \quad \text{or} \quad \xi(t) = \frac{\omega_0 M C \dot{u}_b}{\varepsilon}, \tag{4.26}$$

depending on whether we are talking about the case of internal shot noise or the case of an externally applied random excitation u_b.

★The fluctuational term $\varepsilon\xi(t)$ in equation (4.24) can also be written in other ways by choosing one of the following expressions

$$\xi_0(t), \quad \varepsilon^{1/2}\xi_{1/2}(t), \quad \varepsilon^2\xi_2(t), \quad \text{etc.}$$

The choice of a particular way of writing this term is to a certain extent determined by the intensity of the fluctuations (for larger fluctuations, a smaller power of ε is preferable), but is still rather arbitrary. In fact, nothing prevents us from regarding the fluctuations as either small or large, regardless of our choice of the expression $\varepsilon^r\xi_r(t)$. For an actual self-excited system, the question of which version of the fluctuational term to choose is not particularly important, since the basic behavior of the results obtained always turns out to be the same. However, as will be apparent later, the form $\varepsilon^{1/2}\xi_{1/2}$ has one advantage over all the others, i.e., it is the only form such that the fluctuational scatter of the amplitude remains finite as ε is made to approach zero in the equation

$$\ddot{x} + \omega_0^2 x = \varepsilon\omega_0\left(1 - \frac{4\dot{x}^2}{3\omega_0^2 A_0^2}\right)\dot{x} + \varepsilon^{1/2}\omega_0^2\xi(t). \tag{4.27}$$

while holding the other parameters, including $\xi(t)$, constant. Moreover, despite the fact that the form $\xi_0(t)$ does not explicitly exhibit the smallness of the fluctuations, it does allow us to avoid the somewhat artificial procedure of dividing true fluctuations (e.g., $\omega_0 M I_f$) by ε^r and then multiplying the result by the same power of ε when substituting ξ_r into the equation governing the oscillations.★

Next we give some typical numerical values of the parameters figuring in equation (4.24). Suppose the capacitance, inductance, resistance and mutual inductance in the circuit shown in Figures 12 and 13 have the following values:

$$C = 135 \text{ micromicrofarads}, \quad L = 1.6 \text{ millihenry},$$
$$R = 1 \text{ ohm}, \quad M = 0.18 \text{ millihenry}.$$

Then the resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} \approx 7 \times 10^7 \text{ sec}^{-1}.$$

Moreover, suppose we use the Russian tube type $\Gamma-837$, which has the following d-c operating voltages ($u_{fil} = 12\,v$):

$$E_g = -10\,v, \quad E_a = 400\,v, \quad E_{screen} = 200\,v.$$

The characteristic of this tube is represented by the solid curve AB in Figure 15, and the dashed curve CD shows the approximation (4.22)

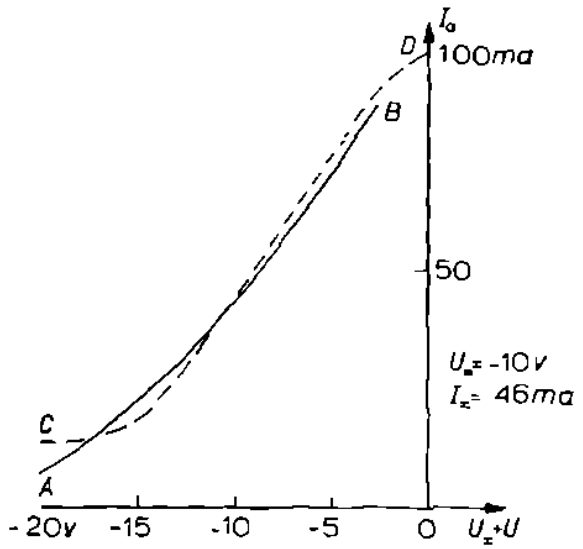

Fig. 15. Actual and approximate grid characteristic of the Γ-837 tube.

corresponding to the numerical values

$$S = 6\,ma/v, \quad \beta = 0.18\,ma/v^2, \quad \gamma = 0.06\,ma/v^3. \tag{4.28}$$

Then, according to (4.25), we have[2]

$$\varepsilon \approx 0.06, \quad A_0 \approx 19\,v.$$

Thus we see that the inequality

$$\varepsilon \ll 1,$$

underlying the theory of self-excited systems which are approximately conservative harmonic oscillators (so-called systems of the *Thomson type*), is well satisfied. The actual experimental value of the amplitude A_0 turns out to be 11 volts instead of 19 volts, which is explained by the fact that the latter value lies in the region where our approximation breaks down (i.e., where the curves AB and CD diverge in Figure 15). A more exact analysis requires the use of a better approximation for large U.

Equation (4.24) without the fluctuational term $\omega_0^2 \varepsilon \xi(t)$ has been studied in detail in the theory of nonlinear oscillations. Naturally, the presence of the random function $\xi(t)$ introduces certain distinctive features, compelling us to supplement the ordinary theory of nonlinear oscillations by introducing probabilistic considerations. In the next two sections, we use the example of equation (4.24) to illustrate the statistical methods of investigating self-excited oscillations in the presence of noise which will be applied later on.

3. Equations in Standard Form and the Simplified Equations

In studying self-excited oscillations, it is convenient to go over from a single second-order equation to two first-order equations which describe the behavior of the amplitude and the phase. As functions of time, the amplitude and phase contain components that are large but slowly varying, in addition to components that are rapidly varying but

[2] In other circuits, one often encounters smaller values of the steepness S, and correspondingly smaller values of the parameter ε.

small. Therefore it can be expected that the time derivatives of the amplitude and phase appearing in the indicated equations will be small.

The concepts of amplitude and phase are intuitively clear, but in trying to define these concepts precisely for oscillations which are not harmonic (although almost so), we encounter a certain arbitrariness that becomes smaller the closer the given oscillations are to being harmonic (cf. Chap. I.7). As the definition uniquely determining the amplitude and phase of the oscillations described by equation (4.3), we choose

$$x = A\cos(\omega_0 t + \varphi), \quad \dot{x} = -\omega_0 A \sin(\omega_0 t + \varphi), \tag{4.29}$$

where ω_0 is the angular frequency figuring in the left-hand side of (4.3). Differentiating the equations

$$A = \left(x^2 + \frac{\dot{x}^2}{\omega_0^2}\right)^{1/2}, \quad \varphi = -\arctan\frac{\dot{x}}{\omega_0 x} - \omega_0 t, \tag{4.30}$$

which give the amplitude and phase explicitly, we find that

$$\dot{A} = \frac{\dot{x}}{\omega_0^2 A}(\ddot{x} + \omega_0^2 x), \quad \dot{\varphi} = -\frac{x}{\omega_0 A^2}(\ddot{x} + \omega_0^2 x), \tag{4.31}$$

or, using (4.3),

$$\dot{A} = \frac{\varepsilon}{\omega_0^2 A} f(x, \dot{x}, t)\dot{x}, \quad \dot{\varphi} = -\frac{\varepsilon}{\omega_0 A^2} f(x, \dot{x}, t)x. \tag{4.32}$$

We must still express the right-hand sides of the equations (4.32) in terms of A and φ. Using (4.29), we find that

$$\dot{A} = \varepsilon G(A, \varphi, t), \quad \dot{\varphi} = \varepsilon H(A, \varphi, t), \tag{4.33}$$

where, for brevity, we introduce the notation

$$\begin{aligned} G(A,\varphi,t) &= -\frac{1}{\omega_0} f(A\cos\Phi, -\omega_0 A\sin\Phi, t)\sin\Phi, \\ H(A,\varphi,t) &= -\frac{1}{\omega_0 A} f(A\cos\Phi, -\omega_0 A\sin\Phi, t)\cos\Phi \end{aligned} \tag{4.34}$$

(Φ denotes the quantity $\omega_0 t + \varphi$). Following Bogoliubov and Mitropolsky,[3] we refer to the system (4.33) as the *equations in standard form*

[3] See N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Non-Linear Oscillations* (translated from the Russian), Gordon and Breach Science Publishers, New York (1961).

(cf. p. I.186). It should be noted that the derivatives in (4.33) are proportional to the small parameter ε. In general, the functions G and H contain a random excitation $\xi(t)$.

Next we find the concrete form of the system (4.33) when the oscillations are described by equation (4.24). According to (4.32), we must multiply the right-hand side of (4.24) first by $\dot{x}/\omega_0^2 A$ and then by $-x/\omega_0 A^2$. This gives

$$\begin{aligned} \dot{A} &= \frac{\varepsilon}{\omega_0 A}\left(\dot{x}^2 - \frac{4\dot{x}^4}{3\omega_0^2 A_0^2}\right) + \frac{\varepsilon}{A}\dot{x}\xi, \\ \dot{\varphi} &= -\frac{\varepsilon}{A^2}\left(1 - \frac{4\dot{x}^2}{3\omega_0^2 A_0^2}\right)x\dot{x} - \frac{\varepsilon\omega_0}{A^2}x\xi. \end{aligned} \tag{4.35}$$

Substituting the expressions (4.29) into (4.35), and going from products and powers of trigonometric functions to functions of multiple angles, we find that

$$\begin{aligned} \dot{A} &= \frac{\varepsilon\omega_0}{2}A\left(1 - \cos 2\Phi - \frac{A^2}{A_0^2}\right. \\ &\qquad \left. + \frac{4A^2}{3A_0^2}\cos 2\Phi - \frac{A^2}{3A_0^2}\cos 4\Phi\right) - \varepsilon\omega_0\xi\sin\Phi, \\ \dot{\varphi} &= \frac{\varepsilon\omega_0}{2}\left[\left(1 - \frac{2A^2}{3A_0^2}\right)\sin 2\Phi + \frac{A^2}{3A_0^2}\sin 4\Phi\right] - \frac{\varepsilon\omega_0}{A}\xi\cos\Phi. \end{aligned} \tag{4.36}$$

Along with certain advantages, the equations in standard form have one drawback, associated with the fact that the functions G and H depend not only on $A(t)$, $\varphi(t)$ and $\xi(t)$, but also explicitly on time through various trigonometric functions of $\omega_0 t$. These functions produce rapid oscillations, and hence will be referred to as "oscillatory." For example, in the first of the equations (4.36), the term

$$A\cos 2\Phi = A\cos(2\omega_0 t + 2\varphi) = A\cos 2\omega_0 t\cos 2\varphi - A\sin 2\omega_0 t\sin 2\varphi \tag{4.37}$$

contains the double-frequency oscillatory functions $\cos 2\omega_0 t$ and $\sin 2\omega_0 t$, in addition to slowly varying functions of A and φ. Our problem is to eliminate the oscillatory functions, thereby obtaining equations which do not contain rapid oscillations. For brevity, the resulting equations will be called the "simplified" equations.

As can be seen from (4.36), rapid oscillations figure not only in regular terms of the type (4.37), but also in the fluctuational terms

$$-\varepsilon\omega_0\xi(t)\sin(\omega_0 t+\varphi), \quad -\frac{\varepsilon\omega_0}{A}\xi(t)\cos(\omega_0 t+\varphi). \tag{4.38}$$

Correspondingly, the problem of eliminating oscillatory functions can be divided into two stages, which can be carried out in either order. Suppose we begin by eliminating oscillations from the nonfluctuational terms. These terms can be isolated by averaging the equations (4.34). Thus we examine the auxiliary equations

$$\dot{A}=\varepsilon\langle G\rangle\equiv\varepsilon G_{av}(A,\varphi), \quad \dot{\varphi}=\varepsilon\langle H\rangle\equiv\varepsilon H_{av}(A,\varphi), \tag{4.39}$$

which in the particular case under consideration take the form

$$\begin{aligned}\dot{A}&=\frac{\varepsilon\omega_0}{2}A\left[1-\frac{A^2}{A_0^2}+\left(\frac{4A^2}{3A_0^2}-1\right)\cos 2\Phi-\frac{A^2}{3A_0^2}\cos 4\Phi\right],\\ \dot{\varphi}&=\frac{\varepsilon\omega_0}{2}\left[\left(1-\frac{2A^2}{3A_0^2}\right)\sin 2\Phi+\frac{A^2}{3A_0^2}\sin 4\Phi\right].\end{aligned} \tag{4.40}$$

Consider the problem of eliminating the oscillatory functions from (4.40). To a first approximation, this can be achieved by simply time-averaging over a period, under the assumption that the amplitude and phase do not change. As a result of applying this familiar method, the equations (4.40) reduce to

$$\dot{A}=\frac{\varepsilon\omega_0}{2}A\left(1-\frac{A^2}{A_0^2}\right), \quad \dot{\varphi}=0. \tag{4.41}$$

These equations, which involve only nonoscillatory terms, will be called the "truncated" equations.

If greater accuracy is required, we must resort to other systematic methods. To higher approximations, corresponding to any desired accuracy, we can obtain nonoscillatory equations by the use of the asymptotic methods developed by Bogoliubov. Here we give a special version of this method, the gist of which consists in introducing the transformation of variables

$$A=A^*+\varepsilon u(A^*,\Phi^*), \quad \Phi=\Phi^*+\varepsilon v(A^*,\Phi^*), \tag{4.42}$$

where

$$\Phi^* = \omega_0 t + \varphi^*.$$

This corresponds to transforming to a new amplitude A^* and phase φ^*, defined somewhat differently than before [see (4.29) and (4.30)]. The functions $u(A^*, \Phi^*)$ and $v(A^*, \Phi^*)$ appearing in the defining equations (4.42) are chosen in such a way that the equations for the new amplitude and phase, which are equivalent to (4.39), contain no oscillatory terms.

The functions u^* and v^* are not found exactly, but rather by successive approximations, i.e., by step-by-step determination of the terms of the expansions

$$\begin{aligned} u(A^*, \Phi^*) &= u_1(A^*, \Phi^*) + \varepsilon u_2(A^*, \Phi^*) + \\ v(A^*, \Phi^*) &= v_1(A^*, \Phi^*) + \varepsilon v_2(A^*, \Phi^*) + \dots \end{aligned} \tag{4.43}$$

At the same time, we find the terms of the nonoscillatory equations

$$\begin{aligned} \dot{A}^* &= \varepsilon G^*_{av}(A^*, \varphi^*) = \varepsilon G^*_1 + \varepsilon^2 G^*_2 + \dots, \\ \dot{\varphi}^* &= \varepsilon H^*_{av}(A^*, \varphi^*) = \varepsilon H^*_1 + \varepsilon^2 H^*_2 + \dots \end{aligned} \tag{4.44}$$

This is done as follows: Substituting (4.42) into (4.39), we find that

$$\begin{aligned} \dot{A}^* + \varepsilon \dot{u} &= \varepsilon G_{av}(A^* + \varepsilon u, \varphi^* + \varepsilon v), \\ \dot{\varphi}^* + \varepsilon \dot{v} &= \varepsilon H_{av}(A^* + \varepsilon u, \varphi^* + \varepsilon v). \end{aligned} \tag{4.45}$$

Bearing in mind that

$$\dot{u} = \frac{\partial u}{\partial A^*} \dot{A}^* + \frac{\partial u}{\partial \Phi^*}(\omega_0 + \dot{\varphi}^*),$$

$$\dot{v} = \frac{\partial v}{\partial A^*} \dot{A}^* + \frac{\partial v}{\partial \Phi^*}(\omega_0 + \dot{\varphi}^*),$$

and substituting (4.44) into (4.45), we obtain the equations

$$\begin{aligned} G^*_{av} + \omega_0 \frac{\partial u}{\partial \Phi^*} &= G_{av}(A^* + \varepsilon u, \varphi^* + \varepsilon v) - \varepsilon \frac{\partial u}{\partial A^*} G^*_{av} - \varepsilon \frac{\partial u}{\partial \Phi^*} H^*_{av}, \\ H^*_{av} + \omega_0 \frac{\partial v}{\partial \Phi^*} &= H_{av}(A^* + \varepsilon u, \varphi^* + \varepsilon v) - \varepsilon \frac{\partial v}{\partial A^*} G^*_{av} - \varepsilon \frac{\partial v}{\partial \Phi^*} H^*_{av}, \end{aligned} \tag{4.46}$$

which serve to define the functions u, v, G^* and H^*. In fact, replacing these functions in (4.46) by their series expansions (4.43) and (4.44)

with respect to ε, and equating terms with identical powers of ε, we find the equations governing the successive approximations, i.e.,

$$\begin{aligned} G_1^* + \omega_0 \frac{\partial u_1}{\partial \Phi^*} &= G_{av}(A^*, \varphi^*), \\ H_1^* + \omega_0 \frac{\partial v_1}{\partial \Phi^*} &= H_{av}(A^*, \varphi^*) \end{aligned} \tag{4.47}$$

in the first approximation, and

$$\begin{aligned} G_2^* + \omega_0 \frac{\partial u_2}{\partial \Phi^*} &= \frac{\partial G_{av}}{\partial A} u_1 + \frac{\partial G_{av}}{\partial \varphi} v_1 - \frac{\partial u_1}{\partial A^*} G_1^* - \frac{\partial u_1}{\partial \Phi^*} H_1^*, \\ H_2^* + \omega_0 \frac{\partial v_2}{\partial \Phi^*} &= \frac{\partial H_{av}}{\partial A} u_1 + \frac{\partial H_{av}}{\partial \varphi} v_1 - \frac{\partial v_1}{\partial A^*} G_1^* - \frac{\partial v_1}{\partial \Phi^*} H_1^* \end{aligned} \tag{4.48}$$

in the second approximation. Each of these equations separates into two parts, i.e., oscillatory terms involving $\partial u/\partial \Phi^*$ and $\partial v/\partial \Phi^*$, and nonoscillatory terms involving G^* and H^*. Together, (4.40) and (4.47) imply

$$G_1^*(A, \varphi) = \frac{\omega_0}{2} A \left(1 - \frac{A^2}{A_0^2}\right), \quad H_1^*(A, \varphi) = 0 \tag{4.49}$$

and

$$\begin{aligned} \frac{\partial u_1(A, \Phi)}{\partial \Phi} &= \frac{A}{2}\left[\left(\frac{4A^2}{3A_0^2} - 1\right) \cos 2\Phi - \frac{A^2}{3A_0^2} \cos 4\Phi\right], \\ \frac{\partial v_1(A, \Phi)}{\partial \Phi} &= \frac{1}{2}\left(1 - \frac{2A^2}{3A_0^2}\right) \sin 2\Phi + \frac{A^2}{6A_0^2} \sin 4\Phi. \end{aligned} \tag{4.50}$$

It follows that

$$\begin{aligned} u_1(A, \Phi) &= \frac{A}{8}\left[2\left(\frac{4A^2}{3A_0^2} - 1\right) \sin 2\Phi - \frac{A^2}{3A_0^2} \sin 4\Phi\right], \\ v_1(A, \Phi) &= -\frac{1}{8}\left[2\left(1 - \frac{2A^2}{3A_0^2}\right) \cos 2\Phi + \frac{A^2}{3A_0^2} \cos 4\Phi\right]. \end{aligned} \tag{4.51}$$

We see at once from (4.49) that the first approximation actually gives the same result as the simple average (4.41). To obtain the second approximation, we have to use (4.48). The functions G_2^* and H_2^* are affected only by the nonoscillatory terms which are not contained in the terms of the form

$$\frac{\partial u}{\partial A^*} G_1^*,$$

i.e.,

$$G_2^* = \text{the nonoscillatory terms in } \frac{\partial G_{av}}{\partial A} u_1 + \frac{\partial G_{av}}{\partial \varphi} v_1,$$

$$H_2^* = \text{the nonoscillatory terms in } \frac{\partial H_{av}}{\partial A} u_1 + \frac{\partial H_{av}}{\partial \varphi} v_1.$$

Taking account of the form of the functions G_{av}, H_{av}, u_1 and v_1, we find that

$$G_2^* = 0, \quad H_2^*(A, \varphi) = -\frac{\omega_0}{16}\left(2 - 4\frac{A^2}{A_0^2} + 3\frac{A^4}{A_0^4}\right). \tag{4.52}$$

Thus, together, (4.42), (4.43) and (4.51) transform the equations (4.40) into the nonoscillatory equations

$$\begin{aligned} \dot{A}^* &= \frac{\varepsilon\omega_0}{2}\left(1 - \frac{A^{*2}}{A_0^2}\right)A^* + O(\varepsilon^3), \\ \dot{\varphi}^* &= -\frac{\varepsilon^2\omega_0}{16}\left(2 - 4\frac{A^{*2}}{A_0^2} + 3\frac{A^{*4}}{A_0^4}\right) + O(\varepsilon^3). \end{aligned} \tag{4.53}$$

By using (4.42), the generated signal $x = A\cos(\omega_0 t + \varphi)$ can easily be expressed in terms of the nonoscillatory amplitude and phase. Retaining only the first correction, we have

$$x = A^*\cos\Phi^* + \varepsilon u_1(A^*, \Phi^*)\cos\Phi^* - \varepsilon A^* v_1(A^*, \Phi^*)\sin\Phi^*, \tag{4.54}$$

and hence, according to (4.51),[4]

$$x = A^*\cos\Phi^* + \frac{\varepsilon}{4}A^*\left(\frac{A^{*2}}{A_0^2} - 1\right)\sin\Phi^* + \frac{\varepsilon}{24}\frac{A^{*3}}{A_0^2}\sin 3\Phi^*. \tag{4.55}$$

[4] It follows from (4.55) that the amplitude and phase A^* and Φ^* do not coincide with the amplitude and phase of the Krylov-Bogoliubov method in the narrow sense, since the latter correspond to the absence of corrections to the fundamental frequency. As is well known, there is a certain arbitrariness in the choice of the nonoscillatory amplitude and phase, which leads to different nonoscillatory equations. Thus, the equations (4.53) can be transformed into the equations of the Krylov-Bogoliubov method by making another change of variables and replacing A^*, φ^* by A_{KB}, φ_{KB}. According to (4.55), the amplitudes A^* and A_{KB} coincide in the first approximation, but, in the same approximation, the phases differ by the quantity

$$\varphi_{KB} - \varphi^* = \frac{\varepsilon}{4}\left(1 - \frac{A^{*2}}{A_0^2}\right).$$

We now return to the complete equations (4.33) and (4.35). The transformation of variables (4.42) and (4.51) eliminates oscillations from the nonfluctuational terms, and carries the equations (4.36) into

$$\begin{aligned}
\dot{A}^* &= \frac{\varepsilon\omega_0}{2}\left(1 - \frac{A^{*2}}{A_0^2}\right)A^* - \varepsilon\omega_0\xi\sin\Phi^* \\
&\quad + \varepsilon^2\omega_0\xi\left[-v_1\cos\Phi^* + \frac{\partial u_1}{\partial A^*}\sin\Phi^* + \frac{\partial u_1}{\partial\Phi^*}\frac{\cos\Phi^*}{A^*}\right], \\
\dot{\varphi}^* &= -\frac{\varepsilon^2\omega_0}{16}\left(2 - 4\frac{A^{*2}}{A_0^2} + 3\frac{A^{*4}}{A_0^4}\right) - \frac{\varepsilon\omega_0}{A^*}\xi\cos\Phi^* \\
&\quad + \frac{\varepsilon^2\omega_0}{A^*}\xi\left[v_1\sin\Phi^* + u_1\frac{\cos\Phi^*}{A^*} + \frac{\partial v_1}{\partial A^*}\sin\Phi^* + \frac{\partial v_1}{\partial\Phi^*}\frac{\cos\Phi^*}{A^*}\right]
\end{aligned} \tag{4.56}$$

[cf. (4.53)], containing oscillations only in the terms involving the random function $\xi(t)$. Although the terms in brackets are formally of order ε^2, it will be seen below that in practice their effect is of the third or fourth order, if the correlation time is not too large. This is explained by the fact that "fluctuational impacts" acting in opposite directions partially compensate each other, thereby diminishing the effect of the fluctuations. Therefore a corresponding modification of our formal scheme for constructing approximations is desirable, in which the equations for the second approximation with simplified nonfluctuational parts become

$$\begin{aligned}
\dot{A}^* &= \frac{\varepsilon\omega_0}{2}\left(1 - \frac{A^{*2}}{A_0^2}\right)A^* - \varepsilon\omega_0\xi\sin\Phi^*, \\
\dot{\varphi}^* &= -\frac{\varepsilon^2\omega_0}{16}\left(2 - 4\frac{A^{*2}}{A_0^2} + 3\frac{A^{*4}}{A_0^4}\right) - \frac{\varepsilon\omega_0}{A^*}\xi\cos\Phi^*.
\end{aligned} \tag{4.57}$$

For notational simplicity, we shall henceforth omit the asterisk indicating the nonfluctuational amplitude and phase, writing (4.57) in the form

$$\dot{A} = \varepsilon G^*(A, \Phi, \xi), \quad \dot{\varphi} = \varepsilon H^*(A, \Phi, \xi), \tag{4.58}$$

where εG^* and εH^* denote the right-hand sides of (4.57). We can even drop the asterisks in the functions G^* and H^*, since this will cause no confusion.

4. Simplification of the Fluctuational Terms

Further simplification of the equations describing the behavior of the amplitude and phase will be based on the assumption that the correlation time τ_{cor} of the random function $\xi(t)$ is comparatively small. Here we do not compare τ_{cor} with the period of the oscillations, but rather with the relaxation time of the amplitude (equal to $1/\varepsilon\omega_0$ in order of magnitude), imposing the requirement that

$$\tau_{cor} \ll \frac{1}{\varepsilon\omega_0}. \tag{4.59}$$

This means that the noise acting on the oscillator has a much greater bandwidth than the generated signal, a condition which is often satisfied in practice (the opposite case will be considered in Chap. 8). The condition (4.59) is perfectly natural from the standpoint of the small-parameter method, since it certainly holds for sufficiently small ε, provided that the rate of the fluctuations is independent of ε.

If the inequality (4.59) holds, we can apply stochastic methods to the present problem, by regarding the evolution of the amplitude in time as a process without aftereffect. According to Chap. I.4, Secs. 8 and 9, starting from (4.58), we can construct a generalized Fokker-Planck equation, involving a series whose terms fall off essentially like powers of the quantity $\varepsilon\omega_0\tau_{cor} \ll 1$ (cf. p. I.99). In the diffusion approximation, where only terms of orders ε and ε^2 are retained, it follows from (4.58) and (I.4.194) that

$$\begin{aligned}
\dot{w}(A,\varphi) &= -\frac{\partial}{\partial A}\left\{\left(\varepsilon\langle G\rangle + \varepsilon^2\int_{-\infty}^{0}\mathbf{K}\left[\frac{\partial G}{\partial A}, G_\tau\right]d\tau + \varepsilon^2\int_{-\infty}^{0}\mathbf{K}\left[\frac{\partial G}{\partial \varphi}, H_\tau\right]d\tau\right)w\right\} \\
&\quad -\frac{\partial}{\partial \varphi}\left\{\left(\varepsilon\langle H\rangle + \varepsilon^2\int_{-\infty}^{0}\mathbf{K}\left[\frac{\partial H}{\partial A}, G_\tau\right]d\tau + \varepsilon^2\int_{-\infty}^{0}\mathbf{K}\left[\frac{\partial H}{\partial \varphi}, H_\tau\right]d\tau\right)w\right\} \\
&\quad +\varepsilon^2\frac{\partial^2}{\partial A^2}\left\{\int_{-\infty}^{0}\mathbf{K}[G, G_\tau]\,d\tau\, w\right\} +
\end{aligned}$$

$$+\varepsilon^2\frac{\partial^2}{\partial A\,\partial\varphi}\left\{\int_{-\infty}^{0}(\mathbf{K}[G,H_\tau]+\mathbf{K}[H,G_\tau])\,d\tau\,w\right\}$$

$$+\varepsilon^2\frac{\partial^2}{\partial\varphi^2}\left\{\int_{-\infty}^{0}\mathbf{K}[H,H_\tau]\,d\tau\,w\right\}, \tag{4.60}$$

where $\varepsilon G=\varepsilon G^*$ and $\varepsilon H=\varepsilon H^*$ denote the right-hand sides of (4.58). Here, in calculating averages and correlation functions, the amplitude A and phase φ figuring in G and H are regarded as fixed quantities and not as random variables. Thus, in calculating a correlation function

$$\mathbf{K}[P,Q]=\langle PQ\rangle-\langle P\rangle\langle Q\rangle,$$

we need only take account of the fluctuational parts of the expressions G^* and H^*, i.e., the terms $-\varepsilon\omega_0\xi\sin\Phi$ and $\varepsilon\omega_0A^{-1}\xi\cos\Phi$ in the case of (4.57).

Because of the presence of oscillatory functions in the fluctuational terms, the right-hand side of equation (4.60) contains oscillatory terms of frequency $2\omega_0$, in addition to smoothly varying terms. Since we are interested in the smooth changes of amplitude and phase, and not in the high-frequency oscillations, this suggests getting rid of the oscillatory terms by performing some kind of an average over a period. In fact, it turns out that such a simplification is permissible, subject to the following reservation: An oscillatory term of order ε not only describes oscillations of the probability density of order ε, but also produces smooth variations of the probability density of order ε^2. Similarly, the oscillatory terms which figure in the terms involving integrals in (4.60), being of order ε^2, produce smooth variations of a still higher order, and hence can be neglected, since equation (4.60) itself is accurate only to the second order. Thus oscillatory terms of the second order can be dropped. The first-order oscillatory terms in $\langle G\rangle$ and $\langle H\rangle$ would lead to considerable complications, if they had not been previously eliminated, since we would then have to take into account their effect on the smooth variations of the probability density in the second order. However, it was just to avoid this difficulty that

we insisted upon preliminary elimination of oscillations from the non-fluctuational part of G and H, as described in the preceding section.

Moreover, it is easily seen from (4.60) that a fluctuational term of order ε (containing $\varepsilon\xi$) actually affects the probability density to order ε^2. But the diffusion equation (4.60) itself is accurate only to order ε^2. In particular, this justifies our previous neglect of terms of the type $-\varepsilon^2\omega_0\xi v_1\cos\Phi$ in (4.56). As far as equation (4.60) is concerned, the retained terms $-\varepsilon\omega_0\xi\sin\Phi$ and $-\varepsilon\omega_0 A^{-1}\xi\cos\Phi$ are effectively of order ε^2, and since this order is the same as the maximum accuracy of (4.60), it makes no sense to look for a third approximation, by using the procedure described on pp. 100–103.

★The situation is different if the random excitation is taken to be of the form $\varepsilon^r\xi_r$ (see p. 95). Then the effective order of the fluctuational terms is $2r$, and we must first eliminate oscillations from $\langle G\rangle$ and $\langle H\rangle$ of orders ε to ε^{2r} inclusive, by constructing successive approximations and redefining the amplitude and phase, i.e., at least $2r$ successive approximations must be made. However, as before, it is pointless to go to an approximation of order higher than $2r$.

In particular, if the fluctuations are chosen to have the form $\varepsilon^{1/2}\xi_{1/2}$, then it is sufficient to calculate G_1^* and H_1^*, and all terms in the equations in standard form will be effectively of the first order, just as in the Fokker-Planck equation. If $r=1$, as assumed previously, it is sufficient to go to the first and second approximations (4.47) and (4.48), and the Fokker-Planck equation is accurate to terms of order ε^2. If the form $\varepsilon^2\xi_2$ is chosen, we must also calculate the functions u_2, v_2 and u_3, v_3. Then these functions are used to define a new amplitude and phase, for which we write equations involving G_1^*, H_1^*, ..., G_4^*, H_4^*, accurate up to terms of order ε^4.★

Having made these general observations, we return to the specific case of equation (4.57), where it only remains to eliminate oscillations due to fluctuations from the integrals appearing in (4.60). Consider one such integral

$$\int_{-\infty}^{0} \mathbf{K}[G, G_\tau]\,d\tau\,.$$

Replacing εG by its random component $-\varepsilon\omega_0\xi\sin\Phi$, we obtain

$$\int_{-\infty}^{0} \mathbf{K}[G, G_\tau]\,d\tau = \omega_0^2 \int_{-\infty}^{0} \langle\xi\xi_\tau\rangle \sin\Phi \sin\Phi_\tau\,d\tau .$$

Using the identity

$$\sin\Phi\sin\Phi_\tau = \tfrac{1}{2}\cos(\Phi_\tau - \Phi) - \tfrac{1}{2}\cos 2\Phi\cos(\Phi_\tau - \Phi) + \tfrac{1}{2}\sin 2\Phi\sin(\Phi_\tau - \Phi),$$

we can transform this equation into

$$\int_{-\infty}^{0} \mathbf{K}[G, G_\tau]\,d\tau = \frac{\omega_0^2}{2}(1 - \cos 2\Phi)\int_{-\infty}^{0} \langle\xi\xi_\tau\rangle \cos\omega_0\tau\,d\tau + \frac{\omega_0^2}{2}\sin 2\Phi\int_{-\infty}^{0} \langle\xi\xi_\tau\rangle \sin\omega_0\tau\,d\tau .$$

In the case of a stationary random process $\xi(t)$, the quantity $\langle\xi\xi_\tau\rangle$ depends only on τ, and the integrals on the right are equal to constants. Therefore the rate of change of the whole expression in time is determined by the trigonometric factors. Separating out the smoothly varying part, we have

$$\int_{-\infty}^{0} \mathbf{K}[G, G_\tau]\,d\tau = \frac{\omega_0^2}{2}\int_{-\infty}^{0} \langle\xi\xi_\tau\rangle \cos\omega_0\tau\,d\tau + \text{oscillatory terms}.$$

Thus in (4.60) we can make the substitution

$$\int_{-\infty}^{0} \mathbf{K}[G, G_\tau]\,d\tau \to \frac{\omega_0^2}{4}\kappa(\omega_0), \tag{4.61}$$

where

$$2\kappa(\omega_0) = 2\int_{-\infty}^{\infty} \langle\xi\xi_\tau\rangle \cos\omega_0\tau\,d\tau = S[\xi;\omega_0]$$

is the spectral density of the noise $\xi(t)$ at the frequency ω_0. Two of the

other integrals can be written in the form

$$\int_{-\infty}^{0} (\mathbf{K}[G, H_\tau] + \mathbf{K}[H, G_\tau])\, d\tau = \frac{\omega_0^2}{4} \int_{-\infty}^{0} \langle \xi\xi_\tau \rangle \sin(\Phi + \Phi_\tau)\, d\tau$$

$$= \frac{\omega_0^2}{A} \sin 2\Phi \int_{-\infty}^{0} \langle \xi\xi_\tau \rangle \cos \omega_0 \tau\, d\tau + \frac{\omega_0^2}{A} \cos 2\Phi \int_{-\infty}^{0} \langle \xi\xi_\tau \rangle \sin \omega_0 \tau\, d\tau, \tag{4.62}$$

and contain purely oscillatory terms. The last integral in the right-hand side of (4.60) is

$$\int_{-\infty}^{0} \mathbf{K}[H, H_\tau]\, d\tau = \frac{\omega_0^2}{A^2} \int_{-\infty}^{0} \langle \xi\xi_\tau \rangle \cos \Phi \cos \Phi_\tau\, d\tau,$$

and can be written in the form

$$\int_{-\infty}^{0} \mathbf{K}[H, H_\tau]\, d\tau = \frac{\omega_0^2}{4A^2} \kappa(\omega_0) + \text{oscillatory terms}, \tag{4.63}$$

by analogy with (4.61).

The integrals in (4.60) involving derivatives can be handled in just the same way. The first of these integrals disappears at once because the fluctuational part of G is independent of A. Of the remaining integrals, which have the form

$$\int_{-\infty}^{0} \mathbf{K}\left[\frac{\partial G}{\partial \varphi}, H_\tau\right] d\tau = \frac{\omega_0^2}{A} \int_{-\infty}^{0} \langle \xi\xi_\tau \rangle \cos \Phi \cos \Phi_\tau\, d\tau,$$

$$\int_{-\infty}^{0} \left\{ \mathbf{K}\left[\frac{\partial H}{\partial A}, G_\tau\right] + \mathbf{K}\left[\frac{\partial H}{\partial \varphi}, H_\tau\right] \right\} d\tau = -\frac{\omega_0^2}{A^2} \int_{-\infty}^{0} \langle \xi\xi_\tau \rangle \sin(\Phi + \Phi_\tau)\, d\tau,$$

only one has a nonoscillatory part:

$$\int_{-\infty}^{0} \mathbf{K}\left[\frac{\partial G}{\partial \varphi}, H_\tau\right] d\tau = \frac{\omega_0^2}{4A} \kappa(\omega_0) + \text{oscillatory terms}. \tag{4.64}$$

Thus, finally, after using (4.61)-(4.64) to eliminate oscillations from the fluctuational terms, the Fokker-Planck equation (4.60) becomes

$$\dot{w} = -\frac{\partial}{\partial A}\left\{\left[\varepsilon\langle G\rangle + \frac{\varepsilon^2\omega_0^2}{4A}\kappa(\omega_0)\right]w\right\} - \frac{\partial}{\partial\varphi}(\varepsilon\langle H\rangle w) + \frac{\varepsilon^2\omega_0^2}{4}\kappa(\omega_0)\left(\frac{\partial^2}{\partial A^2} + \frac{1}{A^2}\frac{\partial^2}{\partial\varphi^2}\right)w. \tag{4.65}$$

Equation (4.65) is equivalent to the following "completely simplified" equations

$$\dot{A} = \frac{\varepsilon\omega_0}{2}\left(1 - \frac{A^2}{A_0^2}\right)A + \frac{\varepsilon^2\omega_0^2}{4A}\kappa(\omega_0) + \varepsilon\omega_0\eta(t), \tag{4.66}$$

$$\dot{\varphi} = -\frac{\varepsilon^2\omega_0}{16}\left(2 - 4\frac{A^2}{A_0^2} + 3\frac{A^4}{A_0^4}\right) + \frac{\varepsilon\omega_0}{A}\zeta(t), \tag{4.67}$$

where $\eta(t)$ and $\zeta(t)$ are identically distributed, mutually independent Gaussian random functions, with zero mean and correlation function

$$\langle\eta\eta_\tau\rangle = \langle\zeta\zeta_\tau\rangle = \frac{\kappa(\omega_0)}{2}\delta(\tau). \tag{4.68}$$

In fact, if we construct the Fokker-Planck equation for the system (4.66) and (4.67), using the general prescription (4.60), we obtain precisely the equation (4.66).

Of course, the simplification of the fluctuational terms which leads from (4.57) to (4.66) and (4.67) can also be accomplished in other ways. One such way is to consider the analogues of the equations (4.57) in rectangular coordinates $A\cos\varphi$ and $A\sin\varphi$. Then A and φ are polar coordinates and the fluctuational terms have a simpler form, independent of the coordinates. We now sketch this alternative approach, which, despite its lack of rigor, is closely related to the method used previously, and to a certain extent gives an intuitive interpretation of some of the terms appearing in equation (4.60).

Thus suppose we write

$$\begin{aligned}\eta_1(t) &= -\xi(t)\sin[\omega_0 t + \varphi(t)],\\ \zeta_1(t) &= -\xi(t)\cos[\omega_0 t + \varphi(t)]\end{aligned} \tag{4.69}$$

in (4.57). Here the phase $\varphi(t)$ is correlated with the values of the random function $\xi(t)$, a fact which greatly complicates the calculation of

the mean values and correlation functions of $\eta_1(t)$ and $\xi_1(t)$. However, we shall assume that the correlation time of the process $\xi(t)$ is so small [cf. (4.49)] that a time shift Δ can be found which simultaneously satisfies the relations

$$\begin{gathered}\Delta \gg \tau_{cor}, \\ |A(t) - A(t-\Delta)| \equiv |A - A_{-\Delta}| \ll A_0, \\ |\varphi(t) - \varphi(t-\Delta)| \equiv |\varphi - \varphi_{-\Delta}| \ll 1. \end{gathered} \tag{4.70}$$

This means that although Δ greatly exceeds the correlation time, the amplitude and phase do not manage to change appreciably during the time Δ. Since the values $A(t-\Delta) \equiv A_{-\Delta}$ and $\varphi(t-\Delta) \equiv \varphi_{-\Delta}$ are close to $A(t)$ and $\varphi(t)$, it is a good approximation to write (4.69) as

$$\begin{aligned}\eta_1 &\approx -\xi \sin(\omega_0 t + \varphi_{-\Delta}) - \xi \cos(\omega_0 t + \varphi_{-\Delta})\Delta\varphi, \\ \zeta_1 &\approx -\xi \cos(\omega_0 t + \varphi_{-\Delta}) + \xi \sin(\omega_0 t + \varphi_{-\Delta})\Delta\varphi, \end{aligned} \tag{4.71}$$

where $\Delta\varphi = \varphi - \varphi_{-\Delta}$. However, $A_{-\Delta}$ and $\varphi_{-\Delta}$, unlike $A(t)$ and $\varphi(t)$, are effectively statistically independent of $\xi(t)$, and hence we can average over ξ and $\varphi_{-\Delta}$ separately. This gives the following expressions for the mean value of (4.71):

$$\begin{aligned}\langle \eta_1 \rangle &\approx -\cos(\omega_0 t + \varphi)\langle \xi \Delta\varphi \rangle, \\ \langle \zeta_1 \rangle &\approx -\sin(\omega_0 t + \varphi)\langle \xi \Delta\varphi \rangle. \end{aligned} \tag{4.72}$$

Next we integrate the equation for the phase, obtaining

$$\Delta\varphi = \varepsilon \int_{t-\Delta}^{t} H\, dt = \varepsilon \int_{-\Delta}^{0} H_\tau\, d\tau,$$

which implies

$$\langle \xi \Delta\varphi \rangle = -\varepsilon\omega_0 \int_{-\Delta}^{0} \left\langle \xi \frac{\xi_\tau}{A_\tau} \cos(\omega_0 t + \omega_0 \tau + \varphi_\tau) \right\rangle d\tau, \tag{4.73}$$

where we have used the form of the fluctuational part of the function H. Once again exploiting the fact that A_τ and φ_τ differ only slightly from the values $A_{\tau-\Delta}$ and $\varphi_{\tau-\Delta}$, which are effectively independent of ξ_τ and ξ, we transform (4.73) into

$$\langle \xi \Delta\varphi \rangle = -\frac{\varepsilon\omega_0}{A} \int_{-\infty}^{0} \langle \xi \xi_\tau \rangle \cos(\omega_0 t + \omega_0 \tau + \varphi)\, d\tau. \tag{4.74}$$

Here we have extended the lower limit of integration to $-\infty$, since the correlation function in the integrand is essentially zero for $|\tau| > \Delta \gg \tau_{cor}$.

Substituting (4.74) into the first of the equations (4.72), we find that

$$\langle \eta_1 \rangle = \frac{\varepsilon\omega_0}{A} \int_{-\infty}^{0} \langle \xi\xi_\tau \rangle \cos(\omega_0 t + \varphi) \cos(\omega_0 t + \omega_0 \tau + \varphi)\, d\tau$$

$$= \frac{\varepsilon\omega_0}{4A} \kappa(\omega_0) + \text{oscillatory terms}. \tag{4.75}$$

Apart from a constant factor, (4.75) coincides with the expression (4.64), which leads to the term

$$\frac{\varepsilon^2\omega_0^2}{4A} \kappa(\omega_0)$$

in (4.65) and (4.66). Thus the appearance of this term in (4.66) is due to the fact that the correlation between $\xi(t)$ and $\varphi(t)$ causes the fluctuational term $\varepsilon\omega_0\eta_1$ to have a nonzero average. Similarly, substituting (4.74) into the second of the equations (4.72), we can easily verify that the mean value $\langle \zeta_1 \rangle$ contains only oscillatory components, which vanish when averaged over a period.

We now find the correlation functions of the zero-mean processes $\eta = \eta_1 - \langle \eta_1 \rangle$ and $\zeta = \zeta_1$. Since in each of the equations (4.71), the presence of the second term on the right affects only the mean values, we can write

$$\eta = -\xi \sin \Phi - \frac{\varepsilon\omega_0}{4A} \kappa(\omega_0) \approx -\xi \sin(\omega_0 t + \varphi_{-\Delta}),$$
$$\zeta = -\xi \cos \Phi \approx -\xi \cos(\omega_0 t + \varphi_{-\Delta}). \tag{4.76}$$

Since the correlation time is much smaller than the relaxation time, the random excitations act as if they were uncorrelated, and the actual correlation function can be replaced by one shaped like a delta function:

$$\langle \eta\eta_\tau \rangle \to \delta(\tau) \int_{-\infty}^{\infty} \langle \eta\eta_\tau \rangle\, d\tau. \tag{4.77}$$

Here we choose the intensity coefficient to be $\int \langle \eta\eta_\tau \rangle d\tau$, since then

integration of both sides of (4.77) with respect to τ gives the same result. Taking account of (4.76), we find that

$$\int_{-\infty}^{\infty} \langle \eta\eta_\tau \rangle \, d\tau = \int_{-\infty}^{\infty} \langle \xi\xi_\tau \rangle \sin(\omega_0 t + \varphi) \sin(\omega_0 t + \omega_0\tau + \varphi_\tau) \, d\tau$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \langle \xi\xi_\tau \rangle \cos(\omega_0\tau + \varphi_\tau - \varphi) \, d\tau + \text{oscillatory terms}$$

$$\approx \frac{\kappa(\omega_0)}{2} + \text{oscillatory terms} \tag{4.78}$$

(since $|\varphi_\tau - \varphi| \ll 1$ for $\tau \sim \tau_{cor}$), and similarly for ζ. It follows at once from (4.77), (4.78) and (4.68) that, as far as their nonoscillatory behavior is concerned, the functions (4.76) coincide with the corresponding functions figuring in (4.66) and (4.67). This justifies the transition from (4.57) to the simplified equations (4.66) and (4.67).

Finally, we note that it is convenient to introduce the dimensionless parameter

$$c = \frac{\varepsilon\omega_0\kappa(\omega_0)}{A_0^2}, \tag{4.79}$$

characterizing the intensity of the fluctuational excitations. In fact, going over to a dimensionless amplitude $a = A/A_0$, we can write the equations (4.66) and (4.67) in the simplified form

$$\begin{aligned} \frac{2}{\varepsilon\omega_0}\dot{a} &= a - a^3 + \frac{c}{2a} + 2\frac{\eta}{A_0}, \\ \frac{2}{\varepsilon\omega_0}\dot{\varphi} &= -\frac{\varepsilon}{8}(2 - 4a^2 + 3a^4) + \frac{2}{a}\frac{\zeta}{A_0}. \end{aligned} \tag{4.80}$$

Supplement

★For completeness, we now describe a general method of eliminating oscillatory terms directly from the stochastic equation. Suppose the latter is of the form

$$\dot{w} = \varepsilon L(t, \varepsilon) w = \varepsilon L_1(t) w + \varepsilon^2 L_2(t) w + \dots, \tag{4.81}$$

where $w = w(z)$, and $L(t, \varepsilon)$ is a linear operator (the method can also be generalized to the case of nonlinear operators). Let the time dependence of $L(t, \varepsilon)$ be due solely to the presence of oscillations, say, to dependence on the functions $\sin k\omega_0 t$ and $\cos k\omega_0 t$. For example, equation (4.60) is of the form (4.81), with $w(z) = w(A, \varphi)$. Suppose it is required to replace (4.81) by an equivalent nonoscillatory equation

$$\dot{w}^* = \varepsilon L^* w^*, \tag{4.82}$$

written in terms of some other function w^*. Then we use the following method, generalizing that described on pp. 100–103. The gist of the method is to make the transformation

$$w = w^* + \varepsilon F[w^*] = w^* + \varepsilon F_1[w^*] + \varepsilon^2 F_2[w^*] + \dots, \tag{4.83}$$

where $F^*[w] \equiv F[w^*, z, t, \varepsilon]$ and the $F_k[w^*, z, t]$ are functionals of w^*, whose dependence on t is oscillatory. Differentiating (4.83) with respect to time and taking account of (4.82), we have

$$\begin{aligned} \dot{w} = {} & \varepsilon L^* w^* + \varepsilon \frac{\partial F_1}{\partial t} + \varepsilon^2 \frac{\partial F_2}{\partial t} + \dots \\ & + \varepsilon \int \frac{\delta F_1}{\delta w^*(z)} \varepsilon L^* w^* \, dz + \varepsilon^2 \int \frac{\delta F_2}{\delta w^*(z)} \varepsilon L^* w^* \, dz + \dots \end{aligned} \tag{4.84}$$

The operator L^* and the transformation (4.82) will be approximated in stages, by successively calculating the terms L_k^* in the expansion

$$\varepsilon L^* = \varepsilon L_1^* + \varepsilon^2 L_2^* + \dots \tag{4.85}$$

and the functionals $F_1, F_2, \dots$ Substituting (4.85) into (4.84), and using (4.81) and (4.83), we obtain the equation

$$\begin{aligned} & \varepsilon L_1^* w^* + \varepsilon^2 L_2^* w^* + \dots + \varepsilon \frac{\partial F_1}{\partial t} + \varepsilon^2 \frac{\partial F_2}{\partial t} + \dots \\ & \quad = \varepsilon L(t, \varepsilon) w^* + \varepsilon^2 L(t, \varepsilon) F_1 + \varepsilon^3 L(t, \varepsilon) F_2 + \dots \\ & \qquad - \varepsilon \int \frac{\delta F_1}{\delta w^*} (\varepsilon L_1^* + \varepsilon^2 L_2^* + \dots) w^* dz \\ & \qquad - \varepsilon^2 \int \frac{\delta F_2}{\delta w^*} (\varepsilon L_1^* + \varepsilon^2 L_2^* + \dots) w^* \, dz - \dots, \end{aligned} \tag{4.86}$$

which can be used to generate the desired successive approximations. Equating terms in (4.86) of the same order in ε, we find that

$$L_1^* w^* + \frac{\partial F_1}{\partial t} = L_1(t) w^*,$$

$$L_2^* w^* + \frac{\partial F_2}{\partial t} = L_2(t) w^* + L_1(t) F_1 - \int \frac{\delta F_1}{\delta w^*} L_1^* w^* \, dz,$$

. . . . . .

The nonoscillatory terms in the right-hand sides belong to $L_k^* w^*$, while the oscillatory terms belong to $\partial F_k / \partial t_k$. Therefore

$$L_1^* w^* = \text{nonoscillatory terms in } L_1(t) w^*,$$

$$\frac{\partial F_1}{\partial t} = \text{oscillatory terms in } L_1(t) w^*,$$

$$L_2^* w^* = \text{nonoscillatory terms in } L_2(t) w^* + L_1(t) F_1 - \int \frac{\delta F_1}{\delta w^*} L_1^* w^* \, dz,$$

$$\frac{\partial F_2}{\partial t} = \text{oscillatory terms in } L_2(t) w^* + L_1(t) F_1 - \int \frac{\delta F_1}{\delta w^*} L_1^* w^* \, dz,$$

. . . . . (4.87)

Using these relations, we can successively determine L_1^*, F_1, L_2^*, F_2, ..., i.e., we can write equation (4.82) and the transformation (4.83) to any accuracy.

We now give two examples illustrating the above method.

Example 1. Suppose (4.81) is a first-order equation of the form

$$\dot{w} = -\frac{\varepsilon\omega_0}{2} \frac{\partial}{\partial A} \left\{ \left[A - \frac{A^3}{A_0^2} + \left(\frac{4A^3}{3A_0^2} - A \right) \cos 2\Phi - \frac{A^3}{3A_0^2} \cos 4\Phi \right] w \right\}$$
$$- \frac{\varepsilon\omega_0}{2} \frac{\partial}{\partial \varphi} \left\{ \left[\left(1 - \frac{2A^2}{3A_0^2} \right) \sin 2\Phi + \frac{A^2}{3A_0^2} \sin 4\Phi \right] w \right\} \equiv \varepsilon L_1(t) w, \tag{4.88}$$

corresponding to (4.40) without fluctuations. In this case, the first two terms of (4.87) are

$$L_1^* w^* = -\frac{\omega_0}{2} \frac{\partial}{\partial A} \left[\left(A - \frac{A^3}{A_0^2} \right) w^* \right],$$

$$\frac{\partial F_1}{\partial t} = -\frac{\omega_0}{2}\frac{\partial}{\partial A}\left\{\left[\left(\frac{4A^3}{3A_0^2} - A\right)\cos 2\Phi - \frac{A^3}{3A_0^2}\cos 4\Phi\right]w^*\right\}$$
$$-\frac{\omega_0}{2}\frac{\partial}{\partial \varphi}\left\{\left[\left(1 - \frac{2A^2}{3A_0^2}\right)\sin 2\Phi + \frac{A^2}{3A_0^2}\sin 4\Phi\right]w^*\right\}.$$

It follows from the last equation that

$$F_1 = -\frac{1}{4}\frac{\partial}{\partial A}\left\{\left[\left(\frac{4A^3}{3A_0^2} - A\right)\sin 2\Phi - \frac{A^3}{6A_0^2}\sin 4\Phi\right]w^*\right\}$$
$$+\frac{1}{4}\frac{\partial}{\partial \varphi}\left\{\left[\left(1 - \frac{2A^2}{3A_0^2}\right)\cos 2\Phi + \frac{A^2}{6A_0^2}\cos 4\Phi\right]w^*\right\}. \tag{4.89}$$

Since

$$\int \frac{\delta F_1}{\delta w^*} L_1^* w^*\, dA\, d\varphi$$

contains no nonoscillatory terms, we find that the third of the equations (4.87) takes the form

$$L_2^* w^* = \text{nonoscillatory terms in } L_1(t) F_1.$$

We must now substitute from (4.88) and (4.89), and afterwards drop oscillatory terms. Obviously, the expression

$$-\frac{\omega_0}{2}\frac{\partial}{\partial A}\left[\left(A - \frac{A^3}{A_0^2}\right)F_1\right]$$

contains only oscillatory terms. Moreover, terms of the type $\cos k\Phi \sin l\Phi$ are oscillatory. Therefore $L_2^* w^*$ equals the nonoscillatory terms in the following expression:

$$-\frac{\omega_0}{8}\frac{\partial}{\partial A}\left[\left(\frac{4A^3}{3A_0^2} - A\right)\cos 2\Phi - \frac{A^3}{3A_0^2}\cos 4\Phi\right]$$
$$\times \frac{\partial}{\partial \varphi}\left[\left(1 - \frac{2A^2}{3A_0^2}\right)\cos 2\Phi + \frac{A^2}{6A_0^2}\cos 4\Phi\right]w^*$$
$$+\frac{\omega_0}{8}\frac{\partial}{\partial \varphi}\left[\left(1 - \frac{2A^2}{3A_0^2}\right)\sin 2\Phi + \frac{A^2}{3A_0^2}\sin 4\Phi\right]$$
$$\times \frac{\partial}{\partial A}\left[\left(\frac{4A^3}{3A_0^2} - A\right)\sin 2\Phi - \frac{A^3}{6A_0^2}\sin 4\Phi\right]w^*$$

$$-\frac{\omega_0}{8}\frac{\partial}{\partial\varphi}\left[\left(1-\frac{2A^2}{3A_0^2}\right)\sin 2\Phi+\frac{A^2}{3A_0^2}\sin 4\Phi\right]$$
$$\times\frac{\partial}{\partial\varphi}\left[\left(1-\frac{2A^2}{3A_0^2}\right)\cos 2\Phi+\frac{A^2}{6A_0^2}\cos 4\Phi\right]w^*.$$

Using the formulas

$$\sin k\Phi\frac{\partial}{\partial\varphi}=\frac{\partial}{\partial\varphi}\sin k\Phi-k\cos k\Phi,$$
$$\cos k\Phi\frac{\partial}{\partial\varphi}=\frac{\partial}{\partial\varphi}\cos k\Phi+k\sin k\Phi,$$

we move the operator $\partial/\partial\varphi$ over to the left. Then, dropping a number of oscillatory terms, we have

$$L_2^*w^*=-\frac{\omega_0}{16}\frac{\partial^2}{\partial A\partial\varphi}\left[\left(\frac{4A^3}{3A_0^2}-A\right)\left(1-\frac{2A^2}{3A_0^2}\right)-\frac{A^5}{18A_0^4}\right]w^*$$
$$+\frac{\omega_0}{16}\frac{\partial}{\partial\varphi}\left[\left(1-\frac{2A^2}{3A_0^2}\right)\frac{\partial}{\partial A}\left(\frac{4A^3}{3A_0^2}-A\right)-\frac{A^2}{3A_0^2}\frac{\partial}{\partial A}\frac{A^3}{6A_0^2}\right]w^*$$
$$+\text{nonoscillatory terms in}$$
$$-\frac{\omega_0}{4}\frac{\partial}{\partial\varphi}\left[\left(1-\frac{2A^2}{3A_0^2}\right)\left(-\cos 2\Phi+\frac{1}{2}\frac{\partial}{\partial\varphi}\sin 2\Phi\right)\right.$$
$$\left.+\frac{2A^2}{3A_0^2}\left(-\cos 4\Phi+\frac{1}{4}\frac{\partial}{\partial\varphi}\sin 4\Phi\right)\right]$$
$$\times\left[\left(1-\frac{2A^2}{3A_0^2}\right)\cos 2\Phi+\frac{A^2}{6A_0^2}\cos 4\Phi\right]w^*.$$

Using the relation

$$f(A)\frac{\partial}{\partial A}=\frac{\partial}{\partial A}f(A)-f'(A)$$

to transform the middle term, and retaining only nonoscillatory expressions in the last term, we finally obtain

$$L_2^*w^*=\frac{\omega_0}{16}\frac{\partial}{\partial\varphi}\left[\left(2-4\frac{A^2}{A_0^2}+3\frac{A^4}{A_0^4}\right)w^*\right].$$

after doing a bit of algebra. In this approximation, the nonoscillatory equation (4.82) has the form

$$\dot{w}^* = \varepsilon L_1^* w^* + \varepsilon^2 L_2^* w^*,$$

and corresponds exactly to the equations (4.53) obtained previously for the nonoscillatory amplitude and phase. Thus the method of eliminating oscillations from the partial differential equation (4.88) is equivalent to the method of eliminating oscillations directly from the ordinary differential equations (4.53), i.e., to the method described on pp. 100–103.

Example 2. Next we consider the equation

$$\dot{w} = \varepsilon L_1^0 w + \varepsilon^{2r} \frac{\partial^2}{\partial A\, \partial \varphi}\left[\frac{\omega_0^2 \kappa(\omega_0)}{2} \frac{\sin 2\Phi}{A} w\right] \equiv \varepsilon L_1^0 w + \varepsilon^{2r} L_{2r}(t) w, \tag{4.90}$$

where L_1^0 is some nonoscillatory operator, and the last term is a typical term like those which will appear in (4.60) if the fluctuational terms in (4.47) are chosen in the form $\varepsilon^r \xi \omega_0 \sin \Phi$, $\varepsilon^r \xi \omega_0 \cos \Phi$. The term in (4.90) comes from the term (4.62) in equation (4.60). Applying (4.87) to this case, we see that $L_1^* = L_1^0$, while all the functions F_1, L_2^*, ..., F_{2r-1}, L_{2r}^* vanish. The first nontrivial equation in (4.87) corresponds to the index $2r$, and is of the form

$$\frac{\partial F_{2r}}{\partial t} = \frac{\omega_0^2 \kappa(\omega_0)}{2} \frac{\partial^2}{\partial A\, \partial \varphi}\left[\frac{\sin 2\Phi}{A} w^*\right],$$

which implies

$$F_{2r} = -\frac{\omega_0 \kappa(\omega_0)}{2} \frac{\partial^2}{\partial A\, \partial \varphi}\left[\frac{\cos 2\Phi}{A} w^*\right]. \tag{4.91}$$

We now find the first nonzero nonoscillatory correction to the nonoscillatory equation

$$\dot{w}^* = \varepsilon L_1^0 w^*$$

due to the function (4.91). With the help of (4.80), it is easy to see that the first nonzero correction $\varepsilon^{4r} L_{4r}^* w^*$ comes from the form $\varepsilon^{4r} L_{2r}(t) F_{2r}$,

and the corresponding equation (4.87) is of the form

$$
\begin{aligned}
L^*_{4r}w^* &= \text{nonoscillatory terms in } L_{2r}(t)F_{2r} \\
&= \text{nonoscillatory terms in} \\
&\qquad -\frac{\omega_0^3}{8}\kappa^2(\omega_0)\frac{\partial^2}{\partial A\partial\varphi}\left[\frac{\sin 2\Phi}{A}\frac{\partial^2}{\partial A\,\partial\varphi}\left(\frac{\cos 2\Phi}{A}w^*\right)\right] \\
&= \text{nonoscillatory terms in} \\
&\qquad \frac{\omega_0^3}{8}\kappa^2(\omega_0)\frac{\partial^2}{\partial A\,\partial\varphi}\left[\frac{2\cos 2\Phi}{A}\frac{\partial}{\partial A}\left(\frac{\cos 2\Phi}{A}w^*\right)\right] \\
&= \frac{\omega_0^3}{8}\kappa^2(\omega_0)\frac{\partial^2}{\partial A\,\partial\varphi}\left[\frac{1}{A}\frac{\partial}{\partial A}\left(\frac{w^*}{A}\right)\right].
\end{aligned}
$$

Therefore the oscillatory terms of order ε^{2r} first affect the nonoscillatory terms to the higher order ε^{4r}. All this is in keeping with what was said on pp. 106–107.★

CHAPTER 5

Methods of Solving the Simplified Equations

1. Amplitude Fluctuations as a Markov Process. The Fokker-Planck Equation

In this chapter, we give a brief summary of various methods of solving the simplified equations, deferring a more detailed treatment until subsequent chapters. We begin by examining the method based on the use of the Fokker-Planck equation. The applicability of this method is restricted only by the condition that the correlation time be small compared to the time constant characterizing the rate of change of the amplitude and phase. This condition, which takes the form (4.59) in the case considered above, allows us to regard the evolution of the amplitude and phase in time as a Markov process and the noise perturbations as a delta-correlated process [after making the substitution (4.77)], and guarantees that the probability densities of amplitude and phase obey the Fokker-Planck equation. The Fokker-Planck method is applicable for both large and small noise intensities. Perhaps the most characteristic feature of the method is that it can be used to find one-dimensional stationary probability distributions and also to solve problems involving the reaching of boundaries, even in cases which are intrinsically nonlinear. However, the determination of correlation functions in the nonlinear case is a much more difficult problem.

For example, consider the first of the equations (4.80), which takes

the form

$$\dot{a} = \frac{\varepsilon\omega_0}{2}\left(a - a^3 + \frac{c}{2a}\right) + \eta', \tag{5.1}$$

in terms of the random function

$$\eta' = \frac{\varepsilon\omega_0}{A_0}\eta,$$

with mean zero and correlation function

$$\langle \eta'\eta'_\tau \rangle = \frac{\varepsilon\omega_0}{2} c\delta(\tau)$$

[recall that $c = \varepsilon\omega_0 A_0^{-2}\kappa(\omega_0)$]. As an intuitional aid, we shall think of the probability density $w(a)$ as the distribution along the a-axis of a foreign substance introduced into a homogeneous medium, and, correspondingly, the evolution of the probability density in time will be regarded as diffusion of this substance. Then a realization of the random function $a(t)$ describes the motion along the a-axis of a "representative point," to be thought of as a kind of "molecule" of the foreign substance, whose trajectories have a very complicated form.

According to (5.1), the mean velocity of the representative point at a given position along the a-axis equals

$$\frac{\varepsilon\omega_0}{2}\left(a - a^3 \quad \frac{c}{2a}\right). \tag{5.2}$$

Multiplying (5.2) by the density $w(a)$ of representative points ("molecules"), we obtain a convection current of such points (a probability current) equal to

$$\frac{\varepsilon\omega_0}{2}\left(a - a^3 + \frac{c}{2a}\right)w(a).$$

Moreover, the concentration has a nonzero gradient $\partial w(a)/\partial a$, which produces a diffusion current proportional to the concentration

gradient. As a result, the total probability current equals[1]

$$G(a) = \frac{\varepsilon\omega_0}{2}\left(a - a^3 + \frac{c}{2a}\right)w - \frac{\varepsilon\omega_0 c}{4}\frac{\partial w}{\partial a}. \tag{5.3}$$

Then, according to (5.3), the equation of continuity

$$\dot{w}(a) = -\frac{\partial G(a)}{\partial a} \tag{5.4}$$

is just the Fokker-Planck equation (I.4.33):

$$\dot{w} = -\frac{\partial}{\partial a}\left[\frac{\varepsilon\omega_0}{2}\left(a - a^3 + \frac{c}{2a}\right)w\right] + \frac{\varepsilon\omega_0 c}{4}\frac{\partial^2 w}{\partial a^2}. \tag{5.5}$$

The one-dimensional Fokker-Planck equation (5.5) can be obtained by integrating both sides of the two-dimensional equation (4.65) with respect to φ.

Using (5.5) to find the time variation of $w(a)$ is difficult when the probability is nonlinear, i.e., when the corresponding equation (5.2) for the random function a is nonlinear (as in the present case). However, the stationary distribution of amplitude can easily be determined by using formula (I.4.49). Any initial probability distribution varies in

---

[1] The coefficients of $\partial w/\partial a$ in (5.3) and $\partial^2 w/\partial a^2$ in (5.5) can be inferred by the following argument: For small values of t, the solution of the equation

$$\dot{w} = -\frac{\partial}{\partial a}(Bw) + D\frac{\partial^2 w}{\partial a^2}$$

[cf. (5.5)] is of the form

$$w \approx \frac{1}{2\sqrt{\pi D t}}\, e^{-a_t^2/4Dt},$$

if $a_t = 0$ for $t = 0$. Therefore the "diffusion coefficient" D is given by

$$D = \lim_{t\to 0}\frac{\langle a_t^2\rangle}{2t},$$

which in the present case is just

$$D = \frac{1}{2}\int_{-\infty}^{\infty}\langle \eta'\eta'_\tau\rangle\, d\tau = \frac{\varepsilon\omega_0 c}{4}.$$

time as described by equation (5.5) and eventually approaches a definite stationary distribution which no longer varies with time. Thus, setting $\dot{w}=0$ in (5.5), we obtain the equation

$$\frac{c}{2}\frac{d^2w}{da^2}-\frac{d}{da}\left[\left(a-a^3+\frac{c}{2a}\right)w\right]=0 \tag{5.6}$$

satisfied by the stationary distribution. Integrating (5.6), we find that

$$\frac{c}{2}\frac{dw}{da}-\left(a-a^3+\frac{c}{2a}\right)w=\text{const}. \tag{5.7}$$

According to (5.7), the probability current (5.3) is the same through all cross sections, a condition which is obviously necessary if the probability is not to accumulate anywhere. However, even more can be said in the present case. The representative points cannot systematically go off to infinity or arrive from infinity, since in practice the size of the amplitude is always bounded. Therefore the probability current vanishes for very large values of the amplitude, which means that the probability current vanishes for all other values of the amplitude:

$$\frac{dw}{da}-\left(2\frac{a-a^3}{c}+\frac{1}{a}\right)w=0.$$

Integrating this equation, we find that

$$w(a)=\frac{a}{N'}\exp\left\{\frac{1}{c}\left\{\left(a^2-\frac{a^4}{2}\right)\right\}\right\}=\frac{2a}{N}e^{-(a^2-1)^2/2c}, \tag{5.8}$$

where N is a constant determined by the normalization condition

$$\int_0^\infty w(a)\,da=1.$$

Evaluation of this integral gives

$$N=\sqrt{2c}\int_{-\frac{1}{\sqrt{2c}}}^{\infty}e^{-y^2}\,dy=\sqrt{\frac{\pi c}{2}}\operatorname{erfc}\left(-\frac{1}{\sqrt{2c}}\right). \tag{5.9}$$

Figure 16 shows the form of the stationary probability density $w(a)$ for several values of the parameter c, which specifies the relative

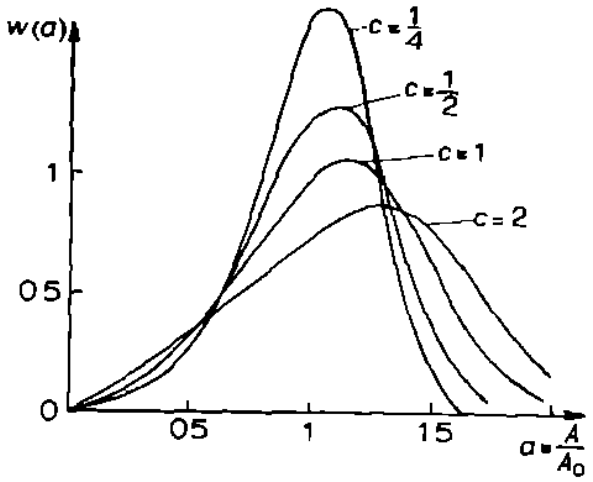

Fig. 16. Stationary amplitude distribution in the presence of broad-band noise.

intensity of the fluctuational excitations. The probability density achieves its maximum value for

$$a_{max} = \frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+2c}}. \tag{5.10}$$

Thus the weaker the noise, the closer the most probable value of the amplitude is to unity, i.e., to the stationary amplitude in the absence of noise. It is clear from the figure that as c decreases, the sharpness of the maximum of the distribution increases rapidly. On the other hand, for large values of c, the maximum of the distribution is not very pronounced, and the amplitude can take values in a wide range.

Starting from the probability density (5.8) and making the change of variable $x=a^2-1$, we find that the moments of the amplitude are given by

$$\langle a^k\rangle = \frac{1}{N}\int_{-1}^{\infty}(1+x)^{k/2}\,e^{-x^2/2c}\,dx. \tag{5.11}$$

To evaluate $\langle a^k\rangle$, it is convenient to use the following recurrence relation, obtained from (5.11) by integration by parts:

$$\langle a^{k+2}\rangle - \langle a^k\rangle = \frac{1}{N}\int_{-1}^{\infty}(1+x)^{k/2}x\,e^{-x^2/2c}\,dx$$

$$= -\frac{c}{N}(1+x)^{k/2}\, e^{-x^2/2c}\Big|_{x=-1}^{x=\infty} + \frac{ck}{2N}\int_{-1}^{\infty}(1+x^2)^{\frac{1}{2}k-1}\, e^{-x^2/2c}\, dx. \tag{5.12}$$

Setting $k=0$ in (5.12), we obtain

$$\langle a^2\rangle = 1 + \frac{c}{N}e^{-1/2c}, \tag{5.13}$$

while if $k>0$, then

$$\langle a^{k+2}\rangle - \langle a^k\rangle = \frac{ck}{2}\langle a^{k-2}\rangle. \tag{5.14}$$

In particular, we have

$$\begin{aligned}\langle a^4\rangle &= \langle a^2\rangle + c = 1 + c + \frac{c}{N}e^{-1/2c},\\ \langle a^6\rangle &= \langle a^4\rangle + 2c\langle a^2\rangle = 1 + 3c + (1+2c)\frac{c}{N}e^{-1/2c}.\end{aligned} \tag{5.15}$$

Before we can use formula (5.14) to find the odd moments, we must first find the quantities $\langle a^{-1}\rangle$ and $\langle a\rangle$. It turns out to be convenient to introduce special notation for the integrals

$$Z_1(y) = 2\int_0^{\infty} e^{-t^4-4yt^2}\, dt, \quad Z_2(y) = 2\int_0^{\infty} e^{-t^4-4yt^2}t^2\, dt, \tag{5.16}$$

in terms of which $\langle a^{-1}\rangle$ and $\langle a\rangle$ are given by the formulas

$$\begin{aligned}\langle a^{-1}\rangle &= \frac{2}{N}e^{-1/2c}\int_0^{\infty}\exp\left\{-\frac{a^4}{2c}+\frac{a^2}{c}\right\}da\\ &= \frac{(2c)^{1/4}}{N}e^{-1/2c}Z_1\left(-\frac{1}{2\sqrt{2c}}\right);\\ \langle a\rangle &= \frac{2}{N}e^{-1/2c}\int_0^{\infty}\exp\left\{-\frac{a^4}{2c}+\frac{a^2}{c}\right\}a^2\, da\\ &= \frac{(2c)^{3/4}}{N}e^{-1/2c}Z_2\left(-\frac{1}{2\sqrt{2c}}\right).\end{aligned} \tag{5.17}$$

It is not hard to show that the integrals (5.16) have the following representations in terms of cylinder functions of fractional order:[2]

$$
\begin{aligned}
Z_1(y) &= y^{1/2} e^{2y^2} K_{1/4}(2y^2) \quad \text{for} \quad y > 0, \\
Z_1(y) &= |y|^{1/2} e^{2y^2} [K_{1/4}(2y^2) + \pi\sqrt{2}\, I_{1/4}(2y^2)] \quad \text{for} \quad y < 0, \\
Z_2(y) &= y^{3/2} e^{2y^2} [K_{3/4}(2y^2) - K_{1/4}(2y^2)] \quad \text{for} \quad y > 0, \\
Z_2(y) &= |y|^{3/2} e^{2y^2} [K_{3/4}(2y^2) + K_{1/4}(2y^2) + \pi\sqrt{2}\, I_{3/4}(2y^2) \\
&\quad + \pi\sqrt{2}\, I_{1/4}(2y^2)] \quad \text{for} \quad y < 0.
\end{aligned}
\tag{5.18}
$$

Numerical values of the functions Z_1 and Z_2 are given in Table 1; the second of these functions will also be needed later (see Chap. 7, Sec. 1). Using these functions, we can examine how the moments (5.17) depend on c. When $c \sim 1$, it turns out that the fluctuational scatter causes the quantities $\langle a \rangle$ and $\langle a^{-1} \rangle$ to differ from unity, but not drastically (by less than a factor of 2).

TABLE 1

x	$Z_1(\sqrt{x/2})$	$Z_1(-\sqrt{x/2})$	$Z_2(\sqrt{x/2})$	$Z_2(-\sqrt{x/2})$
0.1	1.4058	2.6196	0.3399	1.2993
0.3	1.2163	3.8250	0.2381	2.5743
0.6	1.0843	6.3355	0.1777	5.7653
0.9	1.0050	10.4622	0.1453	11.851
1.2	0.9494	17.4873	0.1245	23.498
1.5	0.9067	29.6282	0.1094	45.629
1.8	0.8721	50.8187	0.0984	87.481
2.1	0.8432	87.9723	0.0898	166.23
2.4	0.8191	153.512	0.0822	313.99
2.7	0.7987	269.57	0.0764	590.52
3.0	0.7802	475.61	0.0707	1106.0
3.4	0.7561	1019.17	0.0669	2544.0
3.8	0.7400	2197.4	0.0578	5823.6

In the case $c \ll 1$, when the noise intensity is relatively small, we can use the familiar asymptotic expansions for the cylinder functions

[2] The details are given in the Supplement, p. 144.

figuring in (5.18) to obtain the formulas[3]

$$\begin{aligned} \langle a^{-1}\rangle &= 1 + \tfrac{1}{8}c + O(c^2). \\ \langle a\rangle &= 1 - \tfrac{1}{8}c + O(c^2). \end{aligned} \tag{5.19}$$

According to (5.13), for small c we have

$$\langle a^2\rangle \approx 1 \tag{5.20}$$

to even greater accuracy. Together (5.19) and (5.20) lead to the following approximate expression for the variance of the amplitude, valid for $c \ll 1$:

$$\begin{aligned} \mathbf{D}a = \langle a^2\rangle - \langle a\rangle^2 &= \frac{c}{4} + O(c^2). \\ \langle (A - A_0)^2\rangle = (\langle a^2\rangle - 2\langle a\rangle + 1)A_0^2 &= \frac{c}{4}A_0^2 + O(c^2). \end{aligned} \tag{5.21}$$

For the time being, we confine ourselves to these results, postponing examination of the phase until Chap. 7, Sec. 1.

2. The Linearization Method

The method based on the Fokker-Planck equation and on equations containing delta-correlated processes is applicable only if the correlation time of the noise is small compared to the relaxation time, i.e., if the condition (4.59) is satisfied. The advantage of the linearization method, which we now explain, is that its applicability is not limited by any conditions involving the correlation time. At the same time, compared to the stochastic method just discussed, the linearization method has the disadvantage that the size of the fluctuations must be restricted. In fact, a necessary condition for applying this method is that the fluctuations be small enough to permit us to linearize the basic equations, i.e., to retain only linear terms involving deviations from unperturbed quantities and drop all nonlinear terms.

In studying self-excited oscillations in the linearization approximation, there is no particular need to eliminate oscillatory terms in the

[3] I. M. Ryshik and I. S. Gradstein, *op. cit.*, formulas (6.461.5) and (6.461.6), p. 321.

nonfluctuational parts of the standard equations, since their presence does not prevent us from solving the problem. However, the nonfluctuational parts of the standard equations are still simplified in the same way as before, i.e., we look for a transformation (4.42) carrying the averaged equations (4.39) into the nonoscillatory equations (4.44), which describe smooth variations of amplitude and phase, involving no oscillations.

Denoting smoothly varying averaged trajectories by the label *sm*, we can write (4.44) in the form

$$\dot{A}_{sm} = \varepsilon G^*_{av}(A_{sm}, \varphi_{sm}), \quad \dot{\varphi}_{sm} = \varepsilon H^*_{av}(A_{sm}, \varphi_{sm}). \tag{5.22}$$

Under the transformation (4.42), the standard equations (4.33) go into the equations

$$\dot{A}^* = \varepsilon G^*(A^*, \varphi^*, \xi), \quad \dot{\varphi}^* = \varepsilon H^*(A^*, \varphi^*, \xi) \tag{5.23}$$

[cf. (4.58)], which give the trajectories with fluctuations taken into account. In the linearization approximation, the deviations

$$\delta A = A^* - A_{sm}, \quad \delta\varphi = \varphi^* - \varphi_{sm} \tag{5.24}$$

from the smoothly varying values must be small because of the smallness of the random excitations. Subtracting (5.22) from (5.23) and retaining only linear terms of the expansions in powers of δA and $\delta\varphi$ with averaged coefficients, we obtain the following linear equations, which often turn out to be linear in $\xi(t)$ as well:

$$\begin{aligned}
\delta\dot{A} &= \varepsilon \frac{\partial G^*_{av}(A_{sm}, \varphi_{sm})}{\partial A_{sm}} \delta A + \varepsilon \frac{\partial G^*_{av}(A_{sm}, \varphi_{sm})}{\partial \varphi_{sm}} \delta\varphi \\
&\quad + \varepsilon G^*(A_{sm}, \varphi_{sm}, \xi) - \varepsilon G^*_{av}(A_{sm}, \varphi_{sm}), \\
\delta\dot{\varphi} &= \varepsilon \frac{\partial H^*_{av}(A_{sm}, \varphi_{sm})}{\partial A_{sm}} \delta A + \varepsilon \frac{\partial H^*_{av}(A_{sm}, \varphi_{sm})}{\partial \varphi_{sm}} \delta\varphi \\
&\quad + \varepsilon H^*(A_{sm}, \varphi_{sm}, \xi) - \varepsilon H^*_{av}(A_{sm}, \varphi_{sm}).
\end{aligned} \tag{5.25}$$

Suppose we apply the procedure just described to the equations (4.57), which have exact unsimplified fluctuational parts. In this case,

the linearized equations (5.25) take the form

$$\begin{aligned}\delta\dot{A} &= \frac{\varepsilon\omega_0}{2}\left(1 - 3\frac{A_{sm}^2}{A_0^2}\right)\delta A - \varepsilon\omega_0\xi\sin(\omega_0 t + \varphi_{sm}),\\ \delta\dot{\varphi} &= \frac{\varepsilon^2\omega_0}{4}\left(2\frac{A_{sm}}{A_0^2} - 3\frac{A_{sm}^3}{A_0^4}\right)\delta A - \frac{\varepsilon\omega_0}{A_{sm}}\xi\cos(\omega_0 t + \varphi_{sm}).\end{aligned} \tag{5.26}$$

where A_{sm} and φ_{sm} satisfy the equations

$$\begin{aligned}\dot{A}_{sm} &= \frac{\varepsilon\omega_0}{2}\left(1 - \frac{A_{sm}^2}{A_0^2}\right)A_{sm},\\ \dot{\varphi}_{sm} &= -\frac{\varepsilon^2\omega_0}{16}\left(2 - 4\frac{A_{sm}^2}{A_0^2} + 3\frac{A_{sm}^4}{A_0^4}\right).\end{aligned} \tag{5.27}$$

For example, consider the fluctuational deviations

$$\delta A = A^* - A_0, \quad \delta\varphi = \varphi^* - \varphi_{sm}$$

from the stationary value of the amplitude $A_{sm} = A_0$ and from the phase

$$\varphi_{sm} = -\frac{\varepsilon^2}{16}\omega_0 t + \varphi_0, \tag{5.28}$$

where φ_0 is some initial phase which we assume to be a random variable uniformly distributed in the interval from 0 to 2π. Then the equations (5.26) become

$$\begin{aligned}\delta\dot{A} &= -\varepsilon\omega_0\delta A - \varepsilon\omega_0\xi\sin(\omega_0 t + \varphi_0),\\ \delta\dot{\varphi} &= -\frac{\varepsilon^2\omega_0}{4}\frac{\delta A}{A_0} - \frac{\varepsilon\omega_0}{A_0}\xi\cos(\omega_1 t + \varphi_0),\end{aligned} \tag{5.29}$$

where

$$\omega_1 = \omega_0\left(1 - \frac{\varepsilon^2}{16}\right).$$

The first of the equations (5.29) is a linear differential equation with solution

$$\delta A(t) = -\varepsilon\omega_0\int_{-\infty}^{t} e^{-\varepsilon\omega_0(t-t')}\xi(t')\sin(\omega_1 t' + \varphi_0)\,dt', \tag{5.30}$$

where we have chosen the arbitrary constant of integration in such a way that $\delta A(t)$ is a stationary random process beginning in the remote past. Multiplying (5.30) by the similar expression evaluated at the time $t+\tau$, and then averaging the result, we obtain the correlation function

$$\langle \delta A \delta A_\tau \rangle = \varepsilon^2 \omega_0^2 \int_{-\infty}^{t} \int_{-\infty}^{t+\tau} e^{-\varepsilon\omega_0(2t+\tau-t'-t'')} \times \langle \xi(t')\xi(t'') \sin(\omega_1 t' + \varphi_0) \sin(\omega_1 t'' + \varphi_0) \rangle \, dt' \, dt'' . \tag{5.31}$$

If the noise is stationary and statistically independent of the initial phase φ_0 of the oscillations, then

$$\langle \xi(t')\xi(t'') \sin(\omega_1 t' + \varphi_0) \sin(\omega_1 t'' + \varphi_0) \rangle = \tfrac{1}{2} \langle \xi(t')\xi(t'') \rangle \cos \omega_1 (t' - t''), \tag{5.32}$$

since

$$\begin{aligned} \langle \sin(\omega_1 t' + \varphi_0) \sin(\omega_1 t'' + \varphi_0) \rangle &= \tfrac{1}{2} \langle \cos \omega_1 (t' - t'') \rangle - \tfrac{1}{2} \langle \cos(\omega_1 t' + \omega_1 t'' + 2\varphi_0) \rangle \\ &= \tfrac{1}{2} \cos \omega_1 (t' - t'') . \end{aligned}$$

Substituting (5.32) into (5.31) and making the change of variables

$$t' - t'' + \tau = \sigma , \qquad \frac{t' + t''}{2} = s ,$$

we can write the correlation function in the form

$$\langle \delta A \, \delta A_\tau \rangle = \frac{\varepsilon^2 \omega_0^2}{2} \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{t + \frac{1}{2}\tau - \frac{1}{2}|\sigma|} e^{-\varepsilon\omega_0(2t+\tau-2s)} \times \langle \xi \xi_{\sigma-\tau} \rangle \cos \omega_1 (\sigma - \tau) \, ds .$$

Evaluating the integral with respect to s, we obtain

$$\langle \delta A \, \delta A_\tau \rangle = \frac{\varepsilon\omega_0}{4} \int_{-\infty}^{\infty} e^{-\varepsilon\omega_0|\sigma|} \langle \xi \xi_{\sigma-\tau} \rangle \cos \omega_1 (\sigma - \tau) d\sigma , \tag{5.33}$$

which, in particular, implies

$$\langle(\delta A)^2\rangle = \frac{\varepsilon\omega_0}{2}\int_0^\infty e^{-\varepsilon\omega_0\sigma}\langle\xi\xi_\sigma\rangle\cos\omega_1\sigma\,d\sigma$$

and

$$\int_{-\infty}^{\infty}\langle\delta A\,\delta A_\tau\rangle\,d\tau = \frac{1}{2}\int_{-\infty}^{\infty}\langle\xi\xi_\varrho\rangle\cos\omega_1\varrho\,d\varrho = \frac{1}{2}\kappa(\omega_1). \qquad (5.34)$$

We can also express $\langle\delta A^2\rangle$ in terms of the spectral density of the noise $\xi(t)$:

$$\langle(\delta A)^2\rangle = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\varepsilon^2\omega_0^2}{\varepsilon^2\omega_0^2+(\omega-\omega_1)^2}\kappa(\omega)\,d\omega. \qquad (5.35)$$

The results just obtained are valid subject to the condition that the nonlinear terms can be neglected in linearizing the first of the equations (4.57). This condition reduces to the inequality

$$\langle(\delta A)^2\rangle \ll A_0^2, \qquad (5.36)$$

which implies that

$$\delta A \ll A_0$$

with overwhelming probability. Using (5.35), we can write the criterion (5.36) for the applicability of the linearization method in the form

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\varepsilon^2\omega_0^2}{\varepsilon^2\omega_0^2+(\omega-\omega_1)^2}\kappa(\omega)\,d\omega \ll A_0^2 \quad (\omega_1\approx\omega_0).$$

Approximating the integral on the left, we can write this condition as

$$\frac{1}{4}\int_{\omega_0-\frac{1}{2}\varepsilon\omega_0}^{\omega_0+\frac{1}{2}\varepsilon\omega_0}\kappa(\omega)\,d\omega \ll A_0^2. \qquad (5.37)$$

Unless the noise is very narrow-band, (5.37) can be reduced further to

$$\frac{1}{4}\varepsilon\omega_0\kappa(\omega_0) \ll A_0^2$$

or

$$\frac{c}{4} \ll 1.$$

On the other hand, formula (5.33) is valid for arbitrary correlation times (provided there is no dependence between ξ and φ_0), including small correlation times which satisfy the condition (4.59). In the latter case, the linearization method and the Fokker-Planck method are both applicable, and naturally they both lead to the same result. In fact, when the correlation time is small compared to $1/\varepsilon\omega_0$, the function $\langle \xi \xi_{\tau-\sigma} \rangle$ in (5.33) has a sharp maximum for $\sigma=\tau$, and is appreciably different from zero only in a narrow range near this value. Therefore, in this case, the much more slowly varying function $e^{-\varepsilon\omega_0|\sigma|}$ appearing in the integrand of (5.33) can be replaced by $e^{-\varepsilon\omega_0|\tau|}$, and as a result we obtain the expressions

$$\begin{aligned} \langle \delta A\, \delta A_\tau \rangle &= \frac{\varepsilon\omega_0}{4} e^{-\varepsilon\omega_0|\tau|} \int_{-\infty}^{\infty} \langle \xi \xi_{\sigma-\tau} \rangle \cos \omega_1 (\sigma - \tau)\, d\sigma \\ &\approx \frac{\varepsilon\omega_0}{4} \kappa(\omega_1) e^{-\varepsilon\omega_1|\tau|} \end{aligned} \tag{5.38}$$

and

$$\langle (\delta A)^2 \rangle = \frac{\varepsilon\omega_0}{4} \kappa(\omega_1) \approx \frac{c}{4} A_0^2. \tag{5.39}$$

An exponentially correlated Markov process has a correlation function like (5.38). This is no coincidence, in view of the fact that stochastic methods are applicable whenever $\tau_{cor} \ll 1/\varepsilon\omega_0$. Correspondingly, formulas (5.38) and (5.39) can be obtained by using the Fokker-Planck equation and equations involving delta-correlated random processes. Thus the fluctuational scatter of the amplitude given by (5.39) is the same as that given by the expression (5.21) found earlier.

If only one of the conditions (4.59) and (5.37) is satisfied, then the choice of the method is unique. Thus the general form of the distribution function cannot be obtained by the linearization method, while on the other hand, the general formula (5.33) cannot be derived by using Markov process methods.

Returning to the second of the equations (5.29), we note that the

first term, containing the deviation δA given by formula (5.30), is much smaller than the second term, and hence can be neglected. As a result, we find that the phase shift after time T is

$$\Delta\varphi = \delta\varphi_T - \delta\varphi = -\frac{\varepsilon\omega_0}{A_0}\int_0^T \xi(t)\cos(\omega_1 t + \varphi_0)\,dt. \tag{5.40}$$

It follows that the variance of the phase shift is

$$\langle(\Delta\varphi)^2\rangle = \frac{\varepsilon^2\omega_0^2}{A_0^2}\int_0^T\int_0^T \langle\xi(t')\,\xi(t'')\cos(\omega_1 t' + \varphi_0)\cos(\omega_1 t'' + \varphi_0)\rangle\,dt'\,dt'',$$

which gives

$$\begin{aligned}\langle(\Delta\varphi)^2\rangle &= \frac{\varepsilon^2\omega_0^2}{2A_0^2}\int_0^T\int_0^T \langle\xi\xi_{t'-t''}\rangle\cos\omega_1(t'-t'')\,dt'\,dt'' \\ &= \frac{\varepsilon^2\omega_0^2}{2A_0^2}\int_{-T}^{T}(T-|\tau|)\langle\xi\xi_\tau\rangle\cos\omega_1\tau\,d\tau.\end{aligned} \tag{5.41}$$

under the same assumptions as those leading to (5.32). For large times T, which greatly exceed the correlation time τ_{corr}, the phase shift acquires a "diffusional character," and the dependence of the variance of the phase shift on time approaches a direct proportionality, i.e.,

$$\langle(\Delta\varphi)^2\rangle = D_0 T, \tag{5.42}$$

where

$$D_0 \approx \frac{\varepsilon^2\omega_0^2}{2A_0^2}\int_{-\infty}^{\infty}\langle\xi\xi_\tau\rangle\cos\omega_1\tau\,d\tau = \frac{\varepsilon\omega_0}{2}c. \tag{5.43}$$

The example analyzed in this section shows that in applying the linearization method to linearized equations of the type (5.25), we can make fruitful use of the correlation theory. In just the same way, various other problems can be solved in the linear approximation, some of which will be considered later (see Chap. 6, Sec. 1 and Chap. 9, Sec. 1).

3. The Quasi-Static Method

The apparatus of Markov processes is no longer applicable when the correlation time is comparable to the time constants describing the rate of change of amplitude and phase, i.e., the amplitude relaxation time and the phase relaxation time (provided the latter exists). *A fortiori*, Markov process methods cannot be applied when the correlation time exceeds these time constants. However, suppose the correlation time greatly exceeds the relaxation time T_{rel}, as is the case when the fluctuations change very slowly, so that

$$\tau_{cor} \gg T_{rel}. \tag{5.44}$$

Then (5.44) can be made the basis for applying the so-called *quasi static method*, which is essentially the opposite of the Markov process method.

In the quasi-static method, we simplify the equations in standard form (4.33) by looking for a transformation from A, φ to a new amplitude and phase A^*, φ^* which eliminates oscillatory terms not from the equations (4.39), but rather from the equations

$$\dot{A} = \varepsilon(G)_{\xi=\text{const}}, \quad \dot{\varphi} = \varepsilon(H)_{\xi=\text{const}}, \tag{5.45}$$

which differ from (4.34) in that ξ is regarded as a constant parameter. Apart from this difference, the indicated transformation is found by the same method as before, i.e., it is the same as (4.42) except for the fact that it explicitly depends on ξ. If it is not the process $\xi(t)$ itself which is slowly varying, but rather certain parameters of $\xi(t)$ [like $\varrho(t)$ and $\chi(t)$ in the example given below], then these parameters should be regarded as constants in (5.45).

Applying this transformation to the actual standard equations (4.34), in which $\xi(t)$ [or $\varrho(t)$, $\chi(t)$, ...] vary in time, although slowly, we obtain the complete nonoscillatory equations

$$\dot{A}^* = \varepsilon G^*(A^*, \varphi^*, \xi), \quad \dot{\varphi}^* = \varepsilon H^*(A^*, \varphi^*, \xi). \tag{5.46}$$

If we now impose the condition that the correlation time be so large that "quasi-stable" values of the amplitude and phase manage to be established for every value of $\xi(t)$ [or $\varrho(t)$, $\chi(t)$, ...], then

$$G^*[A^*, \varphi^*, \xi(t)] = 0, \quad H^*[A^*, \varphi^*, \xi(t)] = 0, \tag{5.47}$$

where these equations are obtained by setting $\dot{A}^* = \dot{\varphi}^* = 0$. If the right-hand sides of (5.46) involve the time derivatives $\dot{\xi}$ (or $\dot{\varrho}$, $\dot{\chi}$, ...), then these derivatives can also be set equal to zero (this is justified by the same argument as used to write $\dot{A}^* = \dot{\varphi}^* = 0$).

The equations (5.47) determine the amplitude and phase as zero-memory functions of the noise, i.e., as functions of the value of $\xi(t)$ [or $\varrho(t)$, $\chi(t)$, ...] at the same instant of time. To find the statistical characteristics of the amplitude and phase, it is now sufficient to use the theory of zero-memory nonlinear transformations, presented in Chap. I.8. To obtain more exact results, the zero-memory amplitude and phase determined by (5.47) can be taken as first approximations, and higher approximations, corresponding to small deviations, can then be sought by using (5.46).

To illustrate the quasi-static method, we consider the effect of a narrow-band perturbation

$$\xi(t) = -\varrho(t)\sin[(\omega_0 - \Delta)t + \chi(t)] \tag{5.48}$$

on the vacuum-tube oscillator described by the equations (4.36), confining ourselves to the first approximation, where $\varrho(t)$ and $\chi(t)$, the amplitude and phase of the noise, vary sufficiently slowly. To be explicit, we assume that the amplitude has a Rayleigh distribution (see p. I.192)

$$w(\varrho)\,d\varrho = \frac{\varrho}{\sigma_0^2} e^{-\varrho^2/2\sigma_0^2}\,d\varrho = -\,d\left(e^{-\varrho^2/2\sigma_0^2}\right), \tag{5.49}$$

where σ_0^2 is the variance of (5.48), under the assumption that the initial phase is completely random:

$$\langle \xi^2 \rangle = \tfrac{1}{2}\langle \varrho^2 \rangle = \sigma_0^2.$$

Representing the fluctuational terms in the form

$$-\varepsilon\omega_0\xi\sin\Phi = \frac{\varepsilon\omega_0}{2}\varrho\left[\cos(\chi - \varphi - \Delta t) - \cos(2\omega_0 t - \Delta t + \chi + \varphi)\right],$$

$$-\varepsilon\omega_0\xi\cos\Phi = \frac{\varepsilon\omega_0}{2}\varrho\left[\sin(\chi - \varphi - \Delta t) + \sin(2\omega_0 t - \Delta t + \chi + \varphi)\right] \tag{5.50}$$

(note that $\Delta t = \Delta \cdot t$), we drop the oscillatory terms in (4.36). As a result, we obtain the following "truncated" equations for the first approximation [cf. (4.41)]:

$$\dot{A} = \frac{\varepsilon\omega_0}{2} A\left(1 - \frac{A^2}{A_0^2}\right) + \frac{\varepsilon\omega_0}{2} \varrho \cos(\chi - \psi),$$
$$\dot{\psi} = \Delta + \frac{\varepsilon\omega_0}{2A} \varrho \sin(\chi - \psi) \quad (\psi = \varphi + \Delta t). \tag{5.51}$$

It is clear from (5.51) that the relaxation time of the amplitude is of order $1/\varepsilon\omega_0$, while the mean relaxation time of the phase can be estimated as

$$T_{rel} = \left\langle \frac{2A}{\varepsilon\omega_0 \varrho} \right\rangle \sim \frac{1}{\varepsilon\omega_0} \frac{A_0}{\sigma_0}. \tag{5.52}$$

Thus, in the present case, the conditions for applicability of the quasi-static method take the form

$$\tau_{cor} \gg \frac{1}{\varepsilon\omega_0}, \quad \tau_{cor} \gg \frac{1}{\varepsilon\omega_0} \frac{A_0}{\sigma_0}. \tag{5.53}$$

If these conditions are satisfied, then the amplitude and phase manage to take "quasi-stable" values causing the expressions (5.51) to vanish. Equating these expressions to zero, we obtain

$$\varrho \cos(\chi - \psi) = A\left(\frac{A^2}{A_0^2} - 1\right), \qquad \varrho \sin(\chi - \psi) = -\frac{2A\Delta}{\varepsilon\omega_0}. \tag{5.54}$$

Squaring both equations (5.54) and adding the results, we find that

$$\varrho^2 = A^2\left[\left(\frac{A^2}{A_0^2} - 1\right)^2 + \left(\frac{2\Delta}{\varepsilon\omega_0}\right)^2\right] \equiv A_0^2 f\left(\frac{A^2}{A_0^2}\right). \tag{5.55}$$

Equations (5.54) and (5.55) represent necessary but not sufficient conditions for the existence of stable values of A and ψ, and the stability of these values must still be verified. Let δA and $\delta\psi$ be the small deviations of A and ψ from their stable values. Then, using (5.51), we find the following linearized equations for δA and $\delta\psi$

$$\delta\dot{A} = a_{11}\,\delta A + a_{12}\,\delta\psi,$$
$$\delta\dot{\psi} = a_{21}\,\delta A + a_{22}\,\delta\psi, \tag{5.56}$$

where

$$a_{11} = \varepsilon \frac{\partial G}{\partial A} = -\frac{\varepsilon\omega_0}{2}\left(3\frac{A^2}{A_0^2} - 1\right),$$

$$a_{12} = \varepsilon \frac{\partial G}{\partial \psi} = \frac{\varepsilon\omega_0}{2} Q \sin(\chi - \psi) = -A\Delta,$$

$$a_{21} = \varepsilon \frac{\partial H}{\partial A} = -\frac{\varepsilon\omega_0}{2A^2} Q \sin(\chi - \psi) = \frac{\Delta}{A}, \tag{5.57}$$

$$a_{22} = \varepsilon \frac{\partial H}{\partial \psi} = -\frac{\varepsilon\omega_0}{2A} Q \cos(\chi - \psi) = -\frac{\varepsilon\omega_0}{2}\left(\frac{A^2}{A_0^2} - 1\right)$$

[we also use (5.54)]. The system (5.56) is equivalent to the second-order equation

$$\delta\ddot{A} - (a_{11} + a_{22})\delta\dot{A} + D\,\delta A = 0,$$

where

$$D = a_{11}a_{22} - a_{12}a_{21},$$

from which it is easily seen that a necessary and sufficient condition for stability (i.e., for the deviations to decay in time) is that

$$a_{11} + a_{22} < 0, \quad D > 0.$$

Taking account of (5.57) and using the function defined in (5.55), we can write these conditions in the form

$$\frac{A^2}{A_0^2} > \frac{1}{2}, \quad f'\left(\frac{A^2}{A_0^2}\right) > 0, \tag{5.58}$$

where

$$f'(z) = \frac{df(z)}{dz}.$$

Next, assuming that

$$\frac{2\Delta}{\varepsilon\omega_0} < \frac{1}{2}, \tag{5.59}$$

we investigate the region of possible values of the quantity $z = A^2/A_0^2$. The function $f(z)$ has a positive derivative for $0 < z < z_1$ and for $z > z_2$, where z_1 and $z_2 (> z_1)$ are the roots of the equation $f'(z) = 0$, equal to

$$z_{1,2} = \frac{1}{3}\left[2 \mp \sqrt{1 - 3\left(\frac{2\Delta}{\varepsilon\omega_0}\right)^2}\right]. \tag{5.60}$$

(note that $\Delta t=\Delta\cdot t$), we drop the oscillatory terms in (4.36). As a result, we obtain the following "truncated" equations for the first approximation [cf. (4.41)]:

$$\begin{aligned}\dot{A} &= \frac{\varepsilon\omega_0}{2}A\left(1-\frac{A^2}{A_0^2}\right)+\frac{\varepsilon\omega_0}{2}\varrho\cos(\chi-\psi),\\ \dot{\psi} &= \Delta+\frac{\varepsilon\omega_0}{2A}\varrho\sin(\chi-\psi)\quad(\psi=\varphi+\Delta t).\end{aligned}\tag{5.51}$$

It is clear from (5.51) that the relaxation time of the amplitude is of order $1/\varepsilon\omega_0$, while the mean relaxation time of the phase can be estimated as

$$T_{rel}=\left\langle\frac{2A}{\varepsilon\omega_0\varrho}\right\rangle\sim\frac{1}{\varepsilon\omega_0}\frac{A_0}{\sigma_0}.\tag{5.52}$$

Thus, in the present case, the conditions for applicability of the quasi-static method take the form

$$\tau_{cor}\gg\frac{1}{\varepsilon\omega_0},\quad \tau_{cor}\gg\frac{1}{\varepsilon\omega_0}\frac{A_0}{\sigma_0}.\tag{5.53}$$

If these conditions are satisfied, then the amplitude and phase manage to take "quasi-stable" values causing the expressions (5.51) to vanish. Equating these expressions to zero, we obtain

$$\varrho\cos(\chi-\psi)=A\left(\frac{A^2}{A_0^2}-1\right),\qquad \varrho\sin(\chi-\psi)=-\frac{2A\Delta}{\varepsilon\omega_0}.\tag{5.54}$$

Squaring both equations (5.54) and adding the results, we find that

$$\varrho^2=A^2\left[\left(\frac{A^2}{A_0^2}-1\right)^2+\left(\frac{2\Delta}{\varepsilon\omega_0}\right)^2\right]\equiv A_0^2 f\left(\frac{A^2}{A_0^2}\right).\tag{5.55}$$

Equations (5.54) and (5.55) represent necessary but not sufficient conditions for the existence of stable values of A and ψ, and the stability of these values must still be verified. Let δA and $\delta\psi$ be the small deviations of A and ψ from their stable values. Then, using (5.51), we find the following linearized equations for δA and $\delta\psi$

$$\begin{aligned}\delta\dot{A} &= a_{11}\,\delta A+a_{12}\,\delta\psi,\\ \delta\dot{\psi} &= a_{21}\,\delta A+a_{22}\,\delta\psi,\end{aligned}\tag{5.56}$$

where

$$a_{11} = \varepsilon\frac{\partial G}{\partial A} = -\frac{\varepsilon\omega_0}{2}\left(3\frac{A^2}{A_0^2} - 1\right),$$
$$a_{12} = \varepsilon\frac{\partial G}{\partial \psi} = \frac{\varepsilon\omega_0}{2}\varrho\sin(\chi-\psi) = -A\Delta,$$
$$a_{21} = \varepsilon\frac{\partial H}{\partial A} = -\frac{\varepsilon\omega_0}{2A^2}\varrho\sin(\chi-\psi) = \frac{\Delta}{A}, \tag{5.57}$$
$$a_{22} = \varepsilon\frac{\partial H}{\partial \psi} = -\frac{\varepsilon\omega_0}{2A}\varrho\cos(\chi-\psi) = -\frac{\varepsilon\omega_0}{2}\left(\frac{A^2}{A_0^2} - 1\right)$$

[we also use (5.54)]. The system (5.56) is equivalent to the second-order equation

$$\delta\ddot{A} - (a_{11} + a_{22})\delta\dot{A} + D\,\delta A = 0,$$

where

$$D = a_{11}a_{22} - a_{12}a_{21},$$

from which it is easily seen that a necessary and sufficient condition for stability (i.e., for the deviations to decay in time) is that

$$a_{11} + a_{22} < 0, \quad D > 0.$$

Taking account of (5.57) and using the function defined in (5.55), we can write these conditions in the form

$$\frac{A^2}{A_0^2} > \frac{1}{2}, \quad f'\left(\frac{A^2}{A_0^2}\right) > 0, \tag{5.58}$$

where

$$f'(z) = \frac{df(z)}{dz}.$$

Next, assuming that

$$\frac{2\Delta}{\varepsilon\omega_0} < \frac{1}{2}, \tag{5.59}$$

we investigate the region of possible values of the quantity $z = A^2/A_0^2$. The function $f(z)$ has a positive derivative for $0 < z < z_1$ and for $z > z_2$, where z_1 and $z_2 (> z_1)$ are the roots of the equation $f'(z) = 0$, equal to

$$z_{1,2} = \frac{1}{3}\left[2 \mp \sqrt{1 - 3\left(\frac{2\Delta}{\varepsilon\omega_0}\right)^2}\right]. \tag{5.60}$$

But clearly, because of (5.59), the first of the conditions (5.58) is violated (although the second holds). However, the values $z>z_2$ satisfy both of the conditions (5.58), and hence correspond to stable values of the amplitude and phase. Therefore, in solving equation (5.55), we must take only the larger root $z>z_2$. However, the stable solution just described does not exist for all values of ϱ. In fact, confining ourselves to values $z\geqslant z_2$, we easily see that the function $f(z)$ takes its minimum value $f(z_2)$ for the value z_2. Therefore a stable solution of (5.55) exists only if

$$\varrho > \varrho_1 = A_0[f(z_2)]^{1/2} = A_0 z_2^{1/2}\left[(z_2-1)^2+\left(\frac{2\Delta}{\varepsilon\omega_0}\right)^2\right]^{1/2}. \tag{5.61}$$

Because of (5.49), this inequality holds with probability

$$P_1 = e^{-\varrho_1^2/2\sigma_0^2}. \tag{5.62}$$

In the opposite case, which occurs with probability $1-P_1$, stable values of amplitude and phase are impossible, and the phase difference $\chi-\psi$ grows without limit. Then the terms involving $\cos(\chi-\psi)$ and $\sin(\chi-\psi)$ in (5.51) behave like oscillatory terms in the nontruncated equations of the type (4.40). Neglecting oscillations of amplitude and phase, we can then truncate the equations (5.51), just as was done in the case of (4.40). The result is

$$\dot{A} = \frac{\varepsilon\omega_0}{2}A\left(1-\frac{A^2}{A_0^2}\right), \quad \dot{\psi} = \Delta \quad (\text{or } \dot{\varphi}=0),$$

where the first of these equations has the solution $A=A_0$. Therefore, with probability $1-P_1$ the amplitude takes the value A_0, and with probability P_1 it takes a value $A>A_0\sqrt{z_2}$ given by (5.55). It follows that the amplitude distribution has a delta-function component $(1-P_1)\delta(A-A_0)$, as well as a continuous component $w_{cont}(A)$. To find $w_{cont}(A)$, we substitute (5.55) into (5.49), obtaining

$$\int_{-\infty}^{A} w_{cont}(A_1)\,dA_1 = P_1 - \int_{A}^{\infty} w_{cont}(A_1)\,dA_1$$

$$= P_1 - \exp\left\{-\frac{A^2}{2\sigma_0^2}\left[\left(\frac{A^2}{A_0^2}-1\right)^2+\left(\frac{2\Delta}{\varepsilon\omega_0}\right)^2\right]\right\}.$$

Therefore the complete probability distribution takes the form

$$w(A) = (1 - P_1)\delta(A - A_0) + \frac{A_0}{\sigma_0^2} a\left[3a^4 - 4a^2 + 1 + \left(\frac{2\Delta}{\varepsilon\omega_0}\right)^2\right] \times \exp\left\{-\frac{A_0^2}{2\sigma_0^2} a^2\left[(a^2 - 1)^2 + \left(\frac{2\Delta}{\varepsilon\omega_0}\right)^2\right]\right\}, \tag{5.63}$$

where

$$a \equiv \frac{A}{A_0} \geqslant \sqrt{z_2}.$$

In the special case where $\Delta = 0$, according to (5.60)–(5.62) we have

$$z_2 = 1, \quad \varrho_1 = 0, \quad P_1 = 1.$$

Therefore the delta-function term in (5.62) drops out, and the formula becomes

$$w(a) = \frac{A_0}{\sigma_0^2} a(a^2 - 1)(3a^2 - 1)\exp\left\{-\frac{A_0^2}{2\sigma_0^2} a^2 (a^2 - 1)^2\right\} \quad (a \geqslant 1). \tag{5.64}$$

The form of the distribution (5.64) is shown in Figure 17.

It will be recalled that in formula (5.8), the size of the fluctuational scatter is determined by the parameter c. In the present case, a similar role is played by the parameter σ_0^2/A_0^2. If this parameter is small, i.e.,

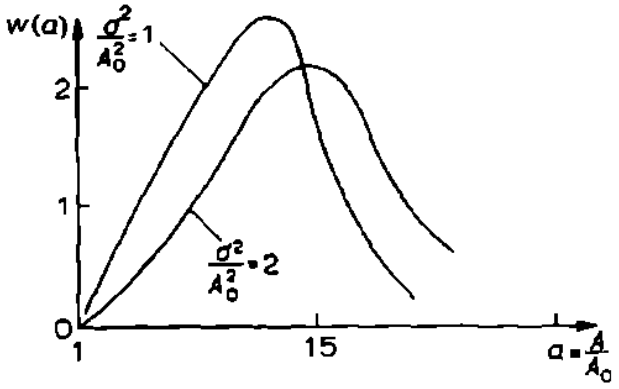

Fig. 17. Probability density of the amplitude for excitation by narrow-band noise

if

$$\frac{\sigma_0^2}{A_0^2} \ll 1, \tag{5.65}$$

then

$$a - 1 \ll 1, \quad \langle a^2 \rangle - 1 \ll 1, \tag{5.66}$$

and hence

$$a^2 - 1 \approx 2(a-1), \quad a^2(a^2-1)^2 \approx 4(a-1)^2, \quad 3a^2 - 1 \approx 2,$$

so that the distribution (5.64) becomes

$$w(A) = 4\frac{A - A_0}{\sigma_0^2} e^{-2(A-A_0)^2/\sigma_0^2}. \tag{5.67}$$

Thus, for small values of the parameter σ_0^2/A_0^2, (5.64) approaches a Rayleigh distribution, shifted along the A-axis. Using (5.67), we find that

$$\langle (A - A_0)^2 \rangle = \frac{\sigma_0^2}{2}. \tag{5.68}$$

Since the inequality (5.65) is the condition for the applicability of the linearization method, we might expect to obtain the relation (5.68) by using this method. Linearizing the equations (5.51) with respect to the deviations $\delta A = A - A_0$ and $\chi - \psi$, which are assumed to be small [a condition for the smallness of $\chi - \psi$ is that the changes of the function $\chi(t)$ be small], we have

$$\delta \dot{A} = -\varepsilon\omega_0 \, \delta A + \frac{\varepsilon\omega_0}{2} \varrho,$$

$$\dot{\psi} = \Delta + \frac{\varepsilon\omega_0}{2A_0} \varrho(t)(\chi - \psi).$$

It is not hard to write the solution of these equations in quadratures. Thus the stationary solution of the first equation is

$$\delta A(t) = \frac{\varepsilon\omega_0}{2} \int_{-\infty}^{t} e^{-\varepsilon\omega_0(t-t')} \varrho(t') \, dt'.$$

In particular, this leads to the correlation function

$$\langle \delta A \, \delta A_\tau \rangle = \frac{\varepsilon\omega_0}{8} \int_{-\infty}^{\infty} e^{-\varepsilon\omega_0|\sigma|} \langle \varrho \varrho_{\sigma-\tau} \rangle \, d\sigma, \tag{5.69}$$

analogous to (5.33). If, besides the assumptions already made, we now assume that the fluctuations change slowly, i.e., $\tau_{cor} \gg 1/\varepsilon\omega_0$ [cf. (5.33)], then (5.69) implies

$$\langle \delta A \, \delta A_\tau \rangle \approx \tfrac{1}{4} \langle \varrho \varrho_\tau \rangle . \tag{5.70}$$

In fact, in the expression (5.69), the exponential factor changes much more rapidly than $\langle \varrho \varrho_{\sigma-\tau} \rangle$ and cuts a narrow interval out of the range of integration, where

$$\sigma \lesssim \frac{1}{\varepsilon\omega} \ll \tau_{cor} \quad \text{and} \quad \langle \varrho \varrho_{\sigma-\tau} \rangle \approx \langle \varrho \varrho_\tau \rangle .$$

Therefore, we can bring $\langle \varrho \varrho_\tau \rangle$ out in front of the integral sign, thereby obtaining (5.70). In particular, setting $\tau = 0$ in (5.70), we find that

$$\langle (\delta A)^2 \rangle = \tfrac{1}{4} \langle \varrho^2 \rangle = \frac{\sigma_0^2}{2} ,$$

which indeed agrees with (5.68).

Thus, just like the stochastic method, the quasi-static method, which is the diametric opposite of the stochastic method, shares a region of applicability with the linearization method.

4. Summary of the Applicability of the Various Methods

Summarizing the results given above, we can assert that whether or not a particular method is applicable depends on the one hand, on the rapidity of change of the fluctuations (in other words, on their spectral composition), and, on the other hand (independently of this rate of change), on their intensity. Here the correlation time must be compared not with the period of the oscillations, but rather with the relaxation time. In spectral language, this means that the bandwidth of the noise perturbations is to be compared with the bandwidth of the generated signal. If the noise spectrum has greater bandwidth $\Delta\omega$ than the spectrum of the self-excited oscillations $\langle \Delta\omega \gg \varepsilon\omega_0 \rangle$, then the apparatus of Markov process theory is applicable, and we can use the Fokker-Planck equation and equations involving delta-correlated processes. On the other hand, if the bandwidth of the noise spectrum is much less than $\varepsilon\omega_0$, then the quasi-static method is applicable, and the

change of amplitude of the self-excited oscillations reduces to a zero-memory dependence on the noise. This applies to the case where the relaxation time of the phase bears the same relation to the correlation time as the relaxation time of the amplitude. Of course, more complicated, combined situations are possible, where, for example, the relaxation time of the phase exceeds τ_{cor}, so that the stochastic method is applicable to the phase equation, while the relaxation time of the amplitude bears the opposite relation to τ_{cor}, so that the amplitude equation can be regarded as quasi-static. For simplicity, however, we shall not consider such cases.

Over and above any dependence on the correlation time, we can use the linearization method if the noise perturbations are small. The relevant linearized equations, whose solutions can easily be written in quadratures, reduce the problem to a linear transformation of a random function, and consequently we can use correlation theory to look for amplitude and phase correlation functions. In many cases, where the random excitation is Gaussian or the correlation time is small, we can also find amplitude and phase distribution laws which are themselves Gaussian. In applying the linearization method, it is not the actual size of the random function $\xi(t)$ that matters, but rather the fluctuational scatter of the self-excited oscillations produced by $\xi(t)$. Strictly speaking, we should compare the standard deviation $\sigma(A)=\sqrt{\overline{\langle(\delta A)^2\rangle}}$ with some "measure of nonlinearity," which might be defined, for example, by the formula

$$\Delta A = \frac{\dfrac{\partial G}{\partial A}}{\dfrac{\partial^2 G}{\partial A^2}}.$$

Then the condition

$$\sigma(A) \ll \Delta A$$

says that the quadratic term $(\partial^2 G/\partial A^2)(\delta A)^2$ plays a small role compared to the linear term $(\partial G/\partial A)\delta A$. However, in most cases, ΔA is of the same order as A_0, and hence $\sigma(A)$ can be compared with A_0 instead of with ΔA. In other words, since order-of-magnitude estimates are sufficient, the sign $\ll$ can be interpreted rather loosely in deciding whether or not the linearization method is applicable. Thus,

for example, we can estimate $\langle(\delta A)^2\rangle$ by using the linearization method, even if, strictly speaking, it is not known to be applicable.

In this chapter, we have considered the case of an additive fluctuational term, which is not involved in the nonlinearity but is merely added to the equation describing the oscillations. Using this example, we obtained the condition (5.36) for the applicability of the linearization method, in the form of the inequalities (5.37) and (5.62). We can also write (5.62) in the form (5.37), since in the case where $\varrho(t)$ and $\chi(t)$ are slowly varying, the whole spectrum is concentrated near ω_0, so that

$$\sigma_0^2 = \frac{1}{\pi}\int_0^\infty \kappa(\omega)\,d\omega \approx \frac{1}{\pi}\int_{\omega_0-\frac{1}{2}\varepsilon\omega_0}^{\omega_0+\frac{1}{2}\varepsilon\omega_0} \kappa(\omega)\,d\omega.$$

Therefore, in the case of an additive fluctuational term, the inequality

$$\frac{1}{4}\int_{\omega_0-\frac{1}{2}\varepsilon\omega_0}^{\omega_0+\frac{1}{2}\varepsilon\omega_0} \kappa(\omega)\,d\omega \ll A_0^2$$

TABLE 2

Rate of change of the noise	Intensity of the noise	
	Small: $\langle(\delta A)^2\rangle \ll A_0^2$	Large: $\langle(\delta A)^2\rangle \sim A_0^2$
Large: $1/\tau_{cor} \gg \varepsilon\omega_0$	1. Linearization method and correlation theory 2. Markov process method	Markov process method
Intermediate: $1/\tau_{cor} \sim \varepsilon\omega_0$	Linearization method and correlation theory	
Small: $1/\tau_{cor} \ll \varepsilon\omega_0$	1. Linearization method and correlation theory 2. Quasi-static method	Quasi-static method

is equivalent to the condition (5.36) for any rate of change of the random excitations.

The various methods for studying self-excited oscillations in the presence of noise, together with conditions for their applicability, are summarized in Table 2, where, as just remarked, for the case of additive noise whose spectrum is concentrated near the fundamental frequency ω_0 and has no sharp maxima at other frequencies, the quantity $\langle(\delta A)^2\rangle$ can be estimated by the expression

$$\frac{1}{4}\int_{\omega_0-\frac{1}{2}\varepsilon\omega_0}^{\omega_0+\frac{1}{2}\varepsilon\omega_0}\kappa(\omega)\,d\omega.$$

In some columns of the table, it is shown that two methods are applicable. At the same time, at the place in the table corresponding to the conditions

$$\tau_{cor}\sim\varepsilon\omega_0,\quad \langle(\delta A)^2\rangle\sim A_0^2,$$

it has unfortunately been necessary to leave an empty space, since sufficiently simple analytic methods suitable for this case are not known.

Supplement

★Here we supply the details of the calculations leading from (5.16) to (5.18). To derive the first of the formulas (5.18), we make the change of variable $t=2y^{1/2}\sinh\xi$, obtaining

$$2\int_0^\infty e^{-t^4-4yt-2y^2}\,dt = 4y^{1/2}\int_0^\infty e^{-2y^2\cosh 4\xi}\cosh\xi\,d\xi$$

$$= y^{1/2}\int_0^\infty e^{-2y^2\cosh\eta}\cosh\frac{\eta}{4}\,d\eta = y^{1/2}K_{1/4}(2y^2),$$

where the last step follows from formula (6.444.1) of Ryshik and Gradstein's handbook (*op. cit.*, p.317).

The third of the formulas (5.18) is easily deduced from the first by

noting that

$$Z_2(y) = -\frac{1}{4}\frac{dZ_1(y)}{dy},$$

and hence

$$\begin{aligned} Z_2(y) &= -\frac{1}{4}\frac{d}{dy}[y^{1/2}e^{2y^2}K_{1/4}(2y^2)] \\ &= -\frac{1}{4}e^{2y^2}\frac{d}{dy}[y^{1/2}K_{1/4}(2y^2)] - y^{3/2}e^{2y^2}K_{1/4}(2y^2). \end{aligned} \tag{5.71}$$

But

$$\frac{d}{dy}[y^{1/2}K_{1/4}(2y^2)] = \frac{dz}{dy}\frac{d}{dz}\left[\left(\frac{z}{2}\right)^{1/4}K_{1/4}(z)\right].$$

where $z=2y^2$. By formulas (6.496.14) and (6.496.16) of the handbook cited, we have

$$\frac{d}{dz}\left[\left(\frac{z}{2}\right)^{1/4}K_{1/4}(z)\right] = -\left(\frac{z}{2}\right)^{1/4}K_{-3/4}(z) = -\left(\frac{z}{2}\right)^{1/4}K_{3/4}(z).$$

Therefore

$$\frac{d}{dy}[y^{1/2}K_{1/4}(2y^2)] = -4y^{3/2}K_{3/4}(2y^2),$$

which when substituted into (5.71) gives the third of the formulas (5.18).

To derive the second and fourth of the formulas (5.18), we have to make the complex continuation of the first and third formulas corresponding to the substitution $y \to e^{\pi i}y_0$, where $\operatorname{Im} y_0 = 0$. Starting from the relations

$$K_p(z) = \frac{\pi}{2}\frac{I_{-p}(z) - I_p(z)}{\sin p\pi},$$

$$I_q(e^{2\pi i}z) = e^{2\pi iq}I_q(z),$$

[see formulas (6.495), (6.455), pp. 328, 319 of Ryshik and Gradstein's handbook], we find that

$$\begin{aligned} (e^{2\pi i}z)^{1/4}K_{1/4}(e^{2\pi i}z) &= \frac{\pi}{\sqrt{2}}z^{1/4}[I_{-1/4}(z) + I_{1/4}(z)] \\ &= z^{1/4}[K_{1/4}(z) + \pi\sqrt{2}I_{1/4}(z)]. \end{aligned} \tag{5.72}$$

and similarly,

$$(e^{2\pi i}z)^{3/4}K_{3/4}(e^{2\pi i}z) = \frac{\pi}{\sqrt{2}}z^{3/4}[I_{-3/4}(z) + I_{3/4}(z)] \qquad (5.73$$
$$= z^{3/4}[K_{3/4}(z) + \pi\sqrt{2}I_{3/4}(z)].$$

According to (5.72),

$$Z_1(e^{\pi i}y_0) = (e^{\pi i}y_0)^{1/2}e^{2y_0^2}K_{1/4}(e^{2\pi i}2y_0^2)$$
$$= y_0^{1/2}e^{2y_0^2}[K_{1/4}(2y_0^2) + \pi\sqrt{2}I_{1/4}(2y_0^2)],$$

which is just the second of the formulas (5.18). Similarly, using both (5.72) and (5.73), we can easily verify the fourth of the formulas (5.18).★

CHAPTER 6

Effect of Weak Internal Noise on an Oscillator

Suppose no external noise acts on an oscillator. Then, even if the operating conditions (e.g., the supply voltages and the temperature) are perfectly stable, the same will not be true of the self-excited oscillations. Thus the constancy of the amplitude and frequency will be destroyed by *internal noise*, i.e., noise arising in the components of the oscillator circuit. The primary source of internal noise is the vacuum tube itself, in which shot noise and cathode ion noise are present. Here we shall consider only the shot noise, since it is predominant at high frequencies and has been studied in greatest detail. Apart from the vacuum tube, thermal noise of thermodynamic origin can also arise in the active circuit resistances. The spectral density of the fluctuating e.m.f. $\mathscr{E}$ arising in a complex impedance $Z(i\omega)$ is given by *Nyquist's formula*

$$S[\mathscr{E};\omega] = 4kT \operatorname{Re} Z(i\omega),$$

and hence is proportional to the active resistance $R = \operatorname{Re} Z$. In a vacuum-tube oscillator like the one shown in Figure 12, with an LC tank circuit and inductive feedback, the main circuit elements are not active resistances. In fact, too high an ohmic resistance in the inductance would destroy the Q of the tank circuit, and the other resistances also play a minor role. Therefore thermal noise is not decisive in such circuits.

Of course, in some cases it may not be permissible to neglect thermal

fluctuations. For example, a resonant circuit whose active resistance contributes important thermal noise may be present in the grid circuit of the oscillator. Thermal noise may also be important in oscillators of the RC and other types, where active resistances appear as basic circuit elements. However, taking account of the thermal noise does not introduce any fundamental difficulties nor does it change our methods of calculation. In principle, thermal noise is even simpler than shot noise, since it is always stationary, with a uniform spectrum. We shall not consider thermal noise further, since its only effect is to increase both the fluctuational scatter of the amplitude and the amount of diffusion of the phase.

1. The Low Intensity of Shot Noise and the Linearized Equations

The considerations of this chapter will be illustrated by the example of the vacuum-tube oscillator shown in Figure 12 and discussed in Chap. 4. The operation of this oscillator is described by equation (4.12), which takes the form (4.20) in terms of the coordinate (4.18). If we introduce the quantity

$$\xi(t) = \frac{\omega_0 M I_f(t)}{\varepsilon} \tag{6.1}$$

[cf. (4.26)] and use the simplest approximation (4.23) for the tube characteristic, then the equation describing the oscillations becomes (4.24), to which everything said in the last two chapters applies.

We begin by estimating the dimensionless parameter

$$c = \frac{\varepsilon \omega_0 \kappa(\omega_0)}{A_0^2} \tag{6.2}$$

for the case of shot noise [cf. (4.79)]. This parameter determines the extent to which the noise affects the operation of the oscillator. Suppose the average anode current is constant and suppose we neglect the effect of both space charge and the cathode-anode electron transit time. Then, according to formula (I.6.40), the shot noise is stationary

and has the correlation function

$$\langle I_f I_{f\tau} \rangle = e\langle I_a \rangle \delta(\tau). \tag{6.3}$$

Therefore it follows from (6.1) and (6.2) that

$$\kappa(\omega) = \frac{\omega_0^2}{\varepsilon^2} M^2 e \langle I_a \rangle$$

and

$$c = \frac{1}{\varepsilon}\left(\frac{\omega_0 M}{A_0}\right)^2 \omega_0 \varepsilon \langle I_a \rangle. \tag{6.4}$$

Next we estimate (6.4) for the example given on p. 96, where A_0, the amplitude of the oscillations of grid voltage, equals 11*v*, while $\langle I_a \rangle$, the average anode current, equals 0.047*a*. Substituting these values (and the others given on p. 96) into (6.4), we find that

$$c = \frac{7}{\varepsilon} \times 10^{-13} = 1.2 \times 10^{-11}. \tag{6.5}$$

The extremely small size of the constant c shows that shot noise has a very small effect, in particular, that the departure of the amplitude from its stationary value is small. Thus we are justified in using the linearization method to study shot noise.

It is interesting to note that the small value (6.5) is obtained despite the appearance of the small parameter ε in the denominator. If we decrease ε, without changing the parameters ω_0, M, A_0, $\langle I_a \rangle$, then, sooner or later, a point is reached when c is no longer small, regardless of the smallness of the shot noise itself. Then the linearization method becomes inapplicable, and the problem can only be solved by applying the apparatus of Markov process theory.

The low intensity of internal noise is perhaps its most characteristic feature. Another characteristic feature is its broad-band nature, i.e., its small correlation time. Naturally, in an anode current undergoing shot noise fluctuations, correlation is preserved for the duration of a current pulse due to a single electron, but the correlation disappears to the extent that electrons in the cathode-anode interspace are replaced by others. Therefore the correlation time of shot noise equals the time of transit of electrons going from the cathode to the anode. This time is ordinarily much smaller than either the relaxation time or

the period, a fact which guarantees the applicability of stochastic methods to the problem of shot noise. Actually, this fact makes the study of internal noise the simplest of all problems associated with noise in oscillators, thereby allowing us to obtain very complete answers to all questions of interest.

In the estimate of the quantity c just given, the average anode current was assumed to be constant, and other simplifying assumptions were made as well. Failure of these assumptions to hold will not change the order of magnitude of c, which will still be much less than unity. Therefore both the linearization method and the stochastic method are applicable to the problem of internal noise in oscillators, even when the specific conditions of the problem are taken into account more fully.

The linearization method is the basic method to be employed below. However, the smallness of the correlation time will be used to further simplify results which would hold in the absence of this condition.

If instead of (4.23), we use the better approximation (4.22) to describe the tube characteristic, (4.20) takes the form

$$\ddot{x} + \omega_0^2 x = \varepsilon\omega_0\left(1 + \frac{\varrho}{2}\frac{\dot{x}}{\omega_0 A_0} - \frac{4}{3}\frac{\dot{x}^2}{\omega_0^2 A_0^2}\right)\dot{x} + \omega_0^2\varepsilon\xi(t), \qquad (6.6)$$

where

$$\varrho = \frac{\omega_0 M\beta A_0}{\varepsilon},$$

which differs from (4.24) only by the presence of the term $\varrho\dot{x}/2\omega_0 A_0$. The equations in standard form corresponding to (6.6) are

$$\dot{a} = \frac{\varepsilon\omega_0}{2}a\left[1 - a^2 - \tfrac{3}{4}\varrho a\sin\Phi + (\tfrac{4}{3}a^2 - 1)\cos 2\Phi\right.$$
$$\left. + \tfrac{1}{4}\varrho a\sin 3\Phi - \tfrac{1}{3}a^2\cos 4\Phi\right] - \frac{\varepsilon\omega_0}{A_0}\xi\sin\Phi,$$

$$\dot{\varphi} = \frac{\varepsilon\omega_0}{2}\left[-\tfrac{1}{4}\varrho a\cos\Phi + (1 - \tfrac{2}{3}a^2)\sin 2\Phi\right.$$
$$\left. + \tfrac{1}{4}\varrho a\cos 3\Phi + \tfrac{1}{3}a^2\sin 4\Phi\right] - \frac{\varepsilon\omega_0}{A_0}\frac{\xi}{a}\cos\Phi, \qquad (6.7)$$

where as usual $a=A/A_0$. Then, by dropping oscillatory terms proportional to sines and cosines of Φ, 2Φ, 3Φ and 4Φ, we obtain the "truncated" equations, which differ from (4.41) only by the presence of the fluctuational terms. Corresponding to these truncated equations, we have the linearized equations

$$\begin{aligned} \delta\dot{a} &= -\varepsilon\omega_0\delta a - \frac{\varepsilon\omega_0}{A_0}\xi\sin(\omega_0 t+\varphi_0), \\ \delta\dot{\varphi} &= -\frac{\varepsilon\omega_0}{A_0}\xi\cos(\omega_0 t+\varphi_0), \end{aligned} \tag{6.8}$$

where, as usual, $\delta a=a-1$, $\delta\varphi=\varphi-\varphi_0$.

Since the fluctuations are very small, the changes in amplitude and phase due to the noise are much smaller than the oscillations of amplitude and phase which invariably accompany nonlinear oscillations and are coherent with the basic frequency. Let us examine the principal oscillations of the latter type. This is done by going over to nonoscillatory values of amplitude and phase, defined by the transformation of variables

$$\begin{aligned} a &= a^* + \varepsilon u(a^*, \Phi^*), \\ \varphi &= \varphi^* + \varepsilon v(a^*, \Phi^*), \end{aligned} \tag{6.9}$$

where $\Phi^*=\omega_0 t+\varphi^*$. To a first approximation, the functions u and v, defined just as on p. 102, are given by

$$\begin{aligned} u(a,\Phi) &= \frac{a}{8}\Big[3\varrho a\cos\Phi + 2\left(\tfrac{1}{3}a^2-1\right)\sin 2\Phi \\ &\qquad - \frac{\varrho a}{3}\cos 3\Phi - \frac{a^2}{3}\sin 4\Phi\Big], \\ v(a,\Phi) &= \frac{1}{8}\Big[-\varrho a\sin\Phi + 2\left(\tfrac{7}{3}a^2-1\right)\cos 2\Phi \\ &\qquad + \frac{\varrho a}{3}\sin 3\Phi - \frac{a^2}{3}\cos 4\Phi\Big]. \end{aligned} \tag{6.10}$$

Then (6.9) transforms (6.7) into the nonoscillatory equations

$$\begin{aligned} \dot{a}^* &= \frac{\varepsilon\omega_0}{2} a^*(1 - a^{*2}) - \frac{\varepsilon\omega_0}{A_0} \xi \sin \Phi^*, \\ \dot{\psi}^* &= -\frac{\varepsilon^2\omega_0}{16}(2 - 4a^{*2} + \tfrac{2}{3}\varrho^2 a^{*2} + 3a^{*4}) - \frac{\varepsilon\omega_0}{A_0} \frac{\ddot{\xi}}{a^*} \cos \Phi^*. \end{aligned} \tag{6.11}$$

The process of making the nonoscillatory components of the amplitude and phase more exact can be extended to higher orders in ε. However, we shall confine ourselves to the most important components, given by (6.10). Using (6.9) and (6.10), we can easily express the generated signal x in terms of the nonoscillatory amplitude and phase [cf. (4.54)]:

$$\begin{aligned} \frac{x}{A_0} &= a \cos \Phi = a^* \cos \Phi^* + \varepsilon u \cos \Phi^* - \varepsilon a^* v \sin \Phi^* \\ &= a^* \cos \Phi^* + \frac{\varepsilon}{4}\varrho a^{*2} - \frac{\varepsilon}{4} a^*(1 - a^{*2}) \sin \Phi^* \\ &\quad + \frac{\varepsilon}{12}\varrho a^{*2} \cos 2\Phi^* + \frac{\varepsilon}{24} a^{*3} \sin 3\Phi^* + O(\varepsilon^2). \end{aligned} \tag{6.12}$$

When the oscillator has established a stationary operating regime, the amplitude a^* is very close to 1, because of the smallness of the noise [this can be seen from the first of the equations (6.11)]. It follows from the second of the equations (6.11) that then the frequency differs only slightly from the frequency

$$\omega_1 = \omega_0 - \frac{\varepsilon^2}{16}\omega_0(1 + \tfrac{2}{3}\varrho^2) \tag{6.13}$$

(to see this, substitute $a^* = 1$). The deviations from the stationary values of the amplitude and frequency will be denoted by

$$\delta a = a^* - 1, \quad \delta\dot{\psi} = \dot{\Phi}^* - \omega_1. \tag{6.14}$$

Bearing in mind that these deviations are small, we deduce the following

linearized equations from (6.11):

$$\begin{aligned}\delta\dot{a} &= -\varepsilon\omega_0\,\delta a - \frac{\varepsilon\omega_0}{A_0}\xi\sin(\omega_1 t+\varphi_0),\\ \delta\dot{\varphi} &= -\frac{\varepsilon^2\omega_0}{4}(1+\tfrac{1}{3}\varrho^2)\delta a - \frac{\varepsilon\omega_0}{A_0}\xi\cos(\omega_1 t+\varphi_0).\end{aligned} \tag{6.15}$$

The small values of the fluctuational deviations also allow us to transform the expression (6.12) for the generated signal into the form

$$\begin{aligned}\frac{x}{A_0} &= \cos\Phi^* + \delta a\cos\Phi^* + \frac{\varepsilon}{2}\varrho\,\delta a + \frac{\varepsilon}{2}\delta a\sin\Phi^* + \frac{\varepsilon}{12}\varrho\cos 2\Phi^*\\ &\quad + \frac{\varepsilon}{6}\varrho\,\delta a\cos 2\Phi^* + \frac{\varepsilon}{24}\sin 3\Phi^* + \frac{\varepsilon}{8}\delta a\sin 3\Phi^* + O(\varepsilon^2)\end{aligned} \tag{6.16}$$

(we omit a nonfluctuational term $\varepsilon\varrho/4$). Retaining only principal terms, we have

$$\frac{x}{A_0} = \cos\Phi^* + \delta a\cos\Phi^* + \frac{\varepsilon}{12}\varrho\cos 2\Phi^* + \frac{\varepsilon}{24}\sin 3\Phi^*. \tag{6.17}$$

In the above derivation of the basic equations (6.15) and (6.17), no assumptions were made about the stationarity of the random excitations $\xi(t)$. Therefore these equations can be used even for the case of nonstationary shot noise.

The very low intensity of shot noise suggests that the corrections of higher order in ε might be larger than the shot noise itself, thereby making it necessary to calculate higher approximations. However, this is not the case, and in fact, even the second-order corrections just calculated are not really necessary, as can be seen by comparing the first and second approximations. It will be noted that the second approximation (6.15) contains two corrections to (6.8): The first is a correction to the average generated frequency ω_0, and the second consists of an extra term in the second equation. Actually, the correction to the frequency exceeds the fluctuational scatter, and the same will be true of the third and higher-order corrections, but these corrections are not of a fluctuational character. Thus the higher-order corrections lead to more precise values of the nonrandom parameters ω_0 and A_0

appearing in the linearized equations, but this is essentially their only effect. The extra term

$$-\varepsilon^2\omega_0(1+\tfrac{1}{3}\varrho^2)\,\delta a$$

in (6.15) plays a small role, since it is proportional to the fluctuations δa, and moreover contains another factor of ε. The analogous terms introduced by the higher approximations will play an even smaller role. Therefore we can actually use the equations (6.8), provided δA and $\delta\dot{\psi}$ are interpreted as deviations from the "true" amplitude and frequency, where the latter may be determined to within less accuracy than the fluctuational scatter itself. In fact, we can omit all calculations of higher-order corrections by the simple expedient of choosing the experimental values as the "true" amplitude and frequency.[1]

2. Shot Noise with Neglect of Periodic Changes of Anode Current

If the average anode current $\langle I_a\rangle$ is constant, the shot noise, and hence the random function $\xi(t)$ appearing in (6.8), will be stationary. If we neglect the time electrons spend in the cathode-anode interspace, the anode current will be a sum

$$I_a(t) = e\sum_j \delta(t-t_j) \tag{6.18}$$

of arbitrarily short pulses, where t_j is the random time of appearance (say, at the anode) of the jth electron. The average current

$$\langle I_a(t)\rangle = en(t) \tag{6.19}$$

and the average number of electrons $n(t)$ in transit will be regarded as constant. Then, according to the second of the formulas (1.6.28), the moment function of the anode current determined by the sum (6.18) is given by

$$\langle I_a(t)I_a(t')\rangle = e^2 f_1(t)\delta(t-t') + e^2 f_2(t,t'), \tag{6.20}$$

[1] The previous inexact values of ω_0 and A_0 can be used in the factors $\varepsilon\omega_0$ and $\varepsilon\omega_0/A_0$ appearing in (6.8), since they are multiplied by the small quantities δa and ξ.

where $f_1(t)$ is the univariate distribution function, equal to $n(t)$, and $f_2(t)$ is the bivariate distribution function (as defined on p. I.144). Similarly, the correlation function of the anode current is given by

$$\mathbf{K}[I_a(t), I_a(t')] = \langle I_f(t) I_f(t') \rangle = e^2 f_1(t) \delta(t-t') + e^2 g_2(t-t'), \tag{6.21}$$

in terms of the correlation function

$$g_2(t,t') = f_2(t,t') - f_1(t) f_1(t'), \tag{6.22}$$

introduced on p. I.147. In the stationary case, (6.22) becomes

$$g_2(t,t') = g_2(t-t') = f_2(t-t') - f_1^2.$$

The function $g_2(t-t')$ describes the correlation between times of flight of the electrons, where $g_2 = 0$ in the absence of any such correlations. If correlations are present, but are sufficiently short-range, we can replace $g_2(t-t')$ by a delta function:

$$g_2(\tau) \to \int_{-\infty}^{\infty} g_2(\tau) d\tau \cdot \delta(\tau). \tag{6.23}$$

Then, according to (6.21), the correlation function of $I_f(t)$ has the "delta-function shape"

$$\langle I_f(t) I_f(t') \rangle = \Gamma^2 e^2 f_1 \delta(t-t') = \Gamma^2 e \langle I_a \rangle \delta(\tau). \tag{6.24}$$

where the coefficient

$$\Gamma^2 = 1 + \frac{1}{f_1} \int_{-\infty}^{\infty} g_2(\tau) d\tau \tag{6.25}$$

is called the *space-charge smoothing factor*. This coefficient is close to unity if the tube operates near saturation. However, if the tube operates in the range where the anode current is limited by space-charge effects, then the space charge reduces the level of the shot noise. In such cases, Γ^2 is less than 1:

$$\Gamma^2 \approx \tfrac{1}{20} \text{ to } \tfrac{1}{10}.$$

Using (6.24), we can easily find the correlation function and spectral density

$$\langle \xi \xi_\tau \rangle = \Gamma^2 e \langle I_a \rangle \left(\frac{\omega_0 M}{\varepsilon} \right)^2 \delta(\tau),$$
$$S[\xi; \omega] = 2\kappa(\omega) = 2\Gamma^2 e \langle I_a \rangle \left(\frac{\omega_0 M}{\varepsilon} \right)^2 \tag{6.26}$$

of the random excitation $\xi(t)$ appearing in the linearized equations (6.8). Just such equations were considered in Chap. 5, Sec. 2 [see (5.29)], where we found the amplitude correlation function (5.33), which, according to (5.38), is proportional to $\kappa(\omega_0)$ in the case of small correlation time. Substituting (6.26) into (5.38), we obtain

$$\langle \delta A \, \delta A_\tau \rangle = \sigma^2 e^{-\varepsilon \omega_0 |\tau|},$$
$$\sigma^2 = \frac{\Gamma^2}{4\varepsilon} e \langle I_a \rangle \omega_0^3 M^2. \tag{6.27}$$

This correlation function corresponds to the following spectral density for the amplitude fluctuations:

$$S[\delta A; \omega] = \frac{4\sigma^2 \varepsilon \omega_0}{\varepsilon^2 \omega_0^2 + \omega^2}. \tag{6.28}$$

Similarly, according to (5.43) and (6.26), the phase diffusion coefficient is

$$D_0 = \tfrac{1}{2}\Gamma^2 e \langle I_a \rangle \left(\frac{\omega_0^2 M}{A_0} \right)^2 = 2 \left(\frac{\sigma}{A_0} \right)^2 \varepsilon \omega_0. \tag{6.29}$$

Because of the small correlation time of the shot noise, the phase shift is of a diffusional character, and we do not take account of deviations from this behavior which occur during intervals of the order of the correlation time.

Using (6.27), we can easily determine the amplitude fluctuations of the current in the inductive branch of the circuit. In fact, according to (4.18), these fluctuations have amplitude A_I connected with A by the relation

$$A_I = \frac{A}{\omega_0 M} = I_0 a, \tag{6.30}$$

where $I_0 = A_0/\omega_0 M$ is the stationary amplitude of oscillations of the current

$$I = I_= + \frac{x}{\omega_0 M}.$$

Therefore it follows from (6.27) that

$$\langle \delta A_t \, \delta A_{t\tau} \rangle = \frac{\Gamma^2}{4\varepsilon} e \langle I_a \rangle \omega_0 e^{-\varepsilon\omega_0|\tau|}. \tag{6.31}$$

It is also convenient to write (6.29) in terms of the stationary amplitude of the current in the inductive branch:

$$D_0 = \tfrac{1}{2}\Gamma^2 e \langle I_a \rangle \frac{\omega_0^2}{I_0^2}. \tag{6.32}$$

The correlation time of the shot noise is much less than all other time constants of the system which are taken into account here. This has the fortunate consequence of allowing us to use the two-dimensional Fokker-Planck equation giving the change in time of the joint distribution of δa and $\delta\varphi$. In the case of linear equations of motion, the Fokker-Planck equation leads to a Gaussian distribution law. Thus we can regard $\delta a(t)$ and $\delta\varphi(t)$ as Gaussian (normal) random processes, with zero means and correlation functions which have already been found. However, we must still find the cross-correlation function of these two processes. Since the random initial phase is uniformly distributed and is statistically independent of $\xi(t)$, we find from (6.26) that

$$\begin{aligned}\langle \xi \sin \Phi \cdot \xi_\tau \cos \Phi_\tau \rangle &= \tfrac{1}{2} \langle \xi \xi_\tau \rangle \langle \sin(\Phi - \Phi_\tau) + \sin(\Phi + \Phi_\tau) \rangle \\ &= \tfrac{1}{2}\Gamma^2 e \langle I_a \rangle \left(\frac{\omega_0 M}{\varepsilon} \right)^2 \delta(\tau) \sin(\varphi - \varphi_\tau) = 0,\end{aligned} \tag{6.33}$$

where the term in $\sin(\Phi + \Phi_\tau)$ vanishes because of the averaging over φ_0, while the last expression, involving $\sin(\varphi - \varphi_\tau)$, vanishes because of the properties of the delta function. It follows from (6.33), at least within the context of the first approximation (6.8), that the amplitude

and phase fluctuations are statistically independent. In fact, multiplying (5.30) by (5.40), averaging the result and using (6.33), we see that $\delta A(t)$ and $\delta\varphi(t)$ are uncorrelated and hence independent, since their joint distribution is Gaussian. Complete information about the processes $\delta a(t)$ and $\delta\varphi(t)$ can be deduced from knowledge of their correlation functions. Using the general rules of Chap. I.3, Sec. 1, we can write the joint distribution function of any number of random values $\delta a(t_1)$, $\delta a(t_2)$, ..., $\delta\varphi(t_1)$, $\delta\varphi(t_2)$, ... for arbitrary t_1, t_2, ... For example, the two-dimensional probability density for δa and δa_τ (where $\tau>0$) is given by

$$w(\delta a, \delta a_\tau) = \frac{1}{2\pi\sigma_1^2}(1-e^{-2\varepsilon\omega_0\tau})^{-1/2} \times \exp\left\{-\frac{(\delta a)^2 - 2e^{-\varepsilon\omega_0\tau}\,\delta a\,\delta a_\tau + (\delta a_\tau)^2}{2\sigma_1^2(1-e^{-2\varepsilon\omega_0\tau})}\right\}, \tag{6.34}$$

where

$$\sigma_1^2 = \frac{\sigma^2}{A_0^2} = \frac{\Gamma^2 e\langle I_a\rangle\omega_0}{4\varepsilon I_0^2}. \tag{6.35}$$

Next we find the correlation function and spectral density of the generated signal, by which we mean the current

$$I = I_= + \frac{x}{\omega_0 M} = \frac{I_0}{A_0}x,$$

or the grid voltage

$$U_g = \frac{\dot{x}}{\omega_0},$$

or even the voltage

$$U_b = \frac{L}{M}U_g = \frac{L}{\omega_0 M}\dot{x}$$

across the inductive branch of the circuit. Since all these quantities are proportional to $x(t)$ or to its derivative, it is sufficient to find the correlation function and spectral density of $x(t)$ itself. Using (6.17) and bearing in mind that the amplitude and phase fluctuations are

independent while $\langle \delta a \rangle = 0$, we find that

$$\frac{\langle x x_\tau \rangle}{A_0^2} = \langle 1 + \delta a\, \delta a_\tau \rangle \langle \cos \Phi^* \cos \Phi_\tau^* \rangle + \frac{\varepsilon}{24} \langle \cos \Phi^* (2\varrho \cos 2\Phi_\tau^* + \sin 3\Phi_\tau^*) \rangle + \frac{\varepsilon}{24} \langle (2\varrho \cos 2\Phi^* + \sin 3\Phi^*) \cos \Phi_\tau^* \rangle + \left(\frac{\varepsilon}{24}\right)^2 \langle (2\varrho \cos 2\Phi^* + \sin 3\Phi^*)(2\varrho \cos 2\Phi_\tau^* + \sin 3\Phi_\tau^*) \rangle . \tag{6.36}$$

Transforming the products of sines and cosines in (6.36) into sums and differences, we average the result over the random initial phase φ_0 appearing in $\Phi^* = \omega_0 t + \varphi_0 + \delta\varphi$. This causes certain terms to drop out, leaving

$$\frac{\langle x x_\tau \rangle}{A_0^2} = \frac{1}{2}(1 + \langle \delta a\, \delta a_\tau \rangle) \cos (\Phi_\tau^* - \Phi^*) + \frac{1}{2}\left(\frac{\varepsilon\varrho}{12}\right)^2 \cos 2(\Phi_\tau^* - \Phi^*) + \frac{1}{2}\left(\frac{\varepsilon}{24}\right)^2 \cos 3(\Phi_\tau^* - \Phi^*). \tag{6.37}$$

Since

$$\Phi_\tau^* - \Phi^* = \omega_0 \tau + \delta\varphi_\tau - \delta\varphi = \omega_0 \tau + \Delta\varphi ,$$

(6.37) becomes

$$\frac{\langle x x_\tau \rangle}{A_0^2} = \frac{1}{2} \operatorname{Re} \left\{ (1 + \langle \delta a\, \delta a_\tau \rangle) \langle e^{i\Delta\varphi} \rangle e^{i\omega_0 \tau} + \left(\frac{\varepsilon\varrho}{12}\right)^2 \langle e^{2i\Delta\varphi} \rangle e^{2i\omega_0 \tau} + \left(\frac{\varepsilon}{24}\right)^2 \langle e^{3i\Delta\varphi} \rangle e^{3i\omega_0 \tau} \right\}. \tag{6.38}$$

The averages

$$\langle e^{i\Delta\varphi} \rangle , \quad \langle e^{2i\Delta\varphi} \rangle , \quad \langle e^{3i\Delta\varphi} \rangle$$

appearing in (6.38) can easily be expressed in terms of the characteristic function of the phase shift, which is of the form

$$\langle e^{iu\Delta\varphi} \rangle = e^{-\frac{1}{2}u^2 D_0 |\tau|}, \tag{6.39}$$

since $\Delta\varphi$ is a Gaussian random variable with variance given by (5.42) and (5.43). Setting $u=1, 2, 3$ in (6.39) and using (6.27), we find that (6.38) becomes

$$\langle x x_\tau \rangle = \frac{A_0^2}{2}\Big[(1+\sigma_1^2 e^{-\varepsilon\omega_0|\tau|})e^{-\frac{1}{4}D_0|\tau|}\cos\omega_0\tau + \left(\frac{\varepsilon Q}{12}\right)^2 e^{-2D_0|\tau|}\cos 2\omega_0\tau + \left(\frac{\varepsilon}{24}\right)^2 e^{-\frac{9}{4}D_0|\tau|}\cos 3\omega_0\tau\Big]. \quad (6.40)$$

Once we know the correlation function, it is an easy matter to find the spectral density, by taking the Fourier transform. The spectral density of the correlation function

$$\sigma^2 e^{-\beta|\tau|}\cos\omega_0\tau$$

has already been calculated in Chap. 1.2, Sec. 1. Using formula (I.2.24) and neglecting the quantity D_0 in the second term, compared to $\varepsilon\omega_0$, we obtain

$$\frac{2}{A_0^2}S[x;\omega] = \frac{D_0}{(\omega-\omega_0)^2+\frac{1}{4}D_0^2} + 2\sigma_1^2\frac{\varepsilon\omega_0}{(\omega-\omega_0)^2+\varepsilon^2\omega_0^2} + \left(\frac{\varepsilon Q}{12}\right)^2\frac{4D_0}{(\omega-2\omega_0)^2+4D_0^2} + \left(\frac{\varepsilon}{24}\right)^2\frac{9D_0}{(\omega-3\omega_0)^2+\frac{81}{4}D_0^2}. \quad (6.41)$$

Because of the conditions $\varepsilon \ll 1$, $D_0 \ll \omega_0$, the spectral density has the line structure shown schematically in Figure 18. In studying the spectral density, we must have an idea of how good an approximation it is to write $D \ll \varepsilon\omega_0$. In the example considered here, we have $\varepsilon \sim 5\times 10^{-2}$, $D_0/\omega_0 \sim 3\times 10^{-13}$ [cf. (5.43) and (6.5)].

Most of the signal power falls into the spectral band lying near the fundamental frequency ω_0. Approximately $(\varepsilon/24)^2$ less power falls into the narrow bands lying near the multiple frequencies $2\omega_0$ and $3\omega_0$. Even less power is associated with the higher harmonics, but to find this power, we must carry out calculations to higher orders in ε.

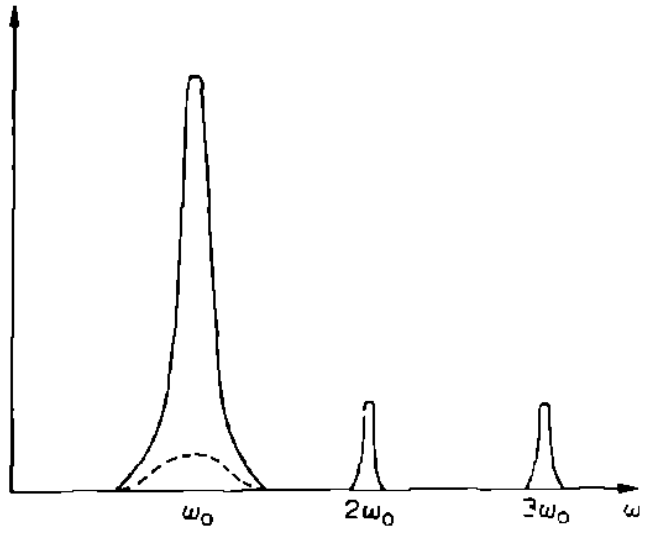

Fig. 18. Spectral structure of the generated signal.

Examining (6.41) in more detail, we see that the spectral density near the fundamental frequency contains two components. The first term in (6.41) describes a narrow-band component, of total power $A_0^2/2$, which is characteristic of the self-excited oscillations and has zero width in the absence of noise. However, because of the phase diffusion, this discrete line is spread out over a bandwidth of order D_0. The second term in (6.41) describes a component which is more broad-band, corresponding to the dashed curve in Figure 18; this component, of bandwidth $\varepsilon\omega_0$, is due to the amplitude fluctuations, and its power

$$\frac{\sigma^2}{2} = \tfrac{1}{8}\Gamma^2 e\langle I_a\rangle \frac{\omega_0 A_0^2}{\varepsilon I_0^2}$$

is less than that of the discrete line corresponding to the fundamental frequency. As for the last two terms in (6.41), they describe the extent to which the discrete lines at the frequencies $2\omega_0$, $3\omega_0$ are spread out. There are also broad-band components at these higher frequencies, but we omit them since they contain even less power.

The spectral densities of the current $I(t)$ and of the voltages U_g and U_b (see p. 158) are expressed in terms of $S[x;\omega]$ by the obvious

formulas

$$S[I-\langle I\rangle;\omega]=\left(\frac{I_0}{A_0}\right)^2 S[x;\omega],$$
$$S[U_g;\omega]=\left(\frac{\omega}{\omega_0}\right)^2 S[x;\omega], \tag{6.42}$$
$$S[U_b;\omega]=\left(\frac{L}{M}\right)^2\left(\frac{\omega}{\omega_0}\right)^2 S[x;\omega].$$

The spectral densities of the voltages U_g and U_b, obtained by differentiating x, differ by having a weak uniform spectral component, i.e., a "white noise component." Although the spectral density of this component is small [equal to $(A_0^2/2\omega_0)^2(D_0+2\sigma_1^2\varepsilon\omega_0+\varepsilon^2 D_0+\ldots)$], it leads to a large (in our approximation, infinite) variance of the voltage fluctuations. To determine the actual size of the variance, we would have to make a more detailed analysis, taking account of the transit time or of parasitic integrating parameters in the circuit.

3. Periodic Nonstationarity of Shot Noise

Since the intensity of shot noise is proportional to the anode current which changes during a period of the oscillations, the noise intensity also varies periodically. Therefore shot noise is actually a nonstationary process. This nonstationarity can be neglected when the a–c component of the anode current is much smaller than its d–c component, and then we can use the results of the preceding section. However, in many practical cases, the self-excited oscillations are quite large, so that it is not permissible to neglect the a–c component of the anode current. For example, in the case considered in Chap. 4, Sec. 2, the anode current has a pronounced pulsating character, causing the size of the anode current and hence the intensity of the shot noise to change by more than a factor of 60 during a single period. Under such conditions, it is natural to expect that taking account of the pulsating character of the noise, i.e., of the periodic changes of its intensity, will lead to appreciable changes in the results found in the preceding chapter.

First we note that formulas (6.18)–(6.25) remain valid even when the average current $\langle I_a(t)\rangle$ and the number of electrons in transit $n(t)$ vary with t, but then the anode current correlation function (6.24) depends not only on the time difference $t-t'$, but also explicitly on t, since $\langle I_a(t)\rangle$ depends on t. Neglecting the effect of transit time, we can express the average anode current (4.9) as a trigonometric series

$$\langle I_a(t)\rangle = F\left[A\cos\left(\Phi+\frac{\pi}{2}\right)\right] = i_0 + i_1\cos\left(\Phi+\frac{\pi}{2}\right) + i_2\cos 2\left(\Phi+\frac{\pi}{2}\right) + i_3\cos 3\left(\Phi+\frac{\pi}{2}\right) + \ldots. \tag{6.43}$$

since the grid voltage is $U_g \approx -A\sin\Phi$. In the case where the tube characteristic is approximated by the polynomial (4.22), the coefficients of the series (6.43) are

$$\begin{aligned} i_0 &= I_= + \frac{\beta}{4}A^2, \quad i_1 = SA - \frac{\gamma}{4}A^3, \\ i_2 &= \frac{\beta}{4}A^2, \quad i_3 = -\frac{\gamma}{12}A^3, \quad i_4 = \ldots = 0. \end{aligned} \tag{6.44}$$

Since the fluctuational deviations from A_0, the stationary amplitude of the grid voltage, are small $(a-1 \ll 1)$, we can write

$$\begin{aligned} i_0 &= I_= + \frac{\beta}{4}A_0^2, \quad i_1 = SA_0 - \frac{\gamma}{4}A_0^3. \\ i_2 &= \frac{\beta}{4}A_0^2, \quad i_3 = -\frac{\gamma}{12}A_0^3, \quad i_4 = \ldots = 0. \end{aligned} \tag{6.45}$$

We now calculate the correlation functions of the random processes

$$\begin{aligned} \zeta_1 &= -\frac{\varepsilon}{A_0}\xi\sin\Phi = \frac{I_f}{I_0}\cos\left(\Phi+\frac{\pi}{2}\right), \\ \zeta_2 &= -\frac{\varepsilon}{A_0}\xi\cos\Phi = -\frac{I_f}{I_0}\sin\left(\Phi+\frac{\pi}{2}\right). \end{aligned} \tag{6.46}$$

Unlike the stationary case considered previously, the fluctuations of anode current are no longer independent of the phase of the oscillations, and hence formula (5.32) is no longer valid. Thus we must

average the product $I_f \sin\Phi\, I_{f\tau} \sin\Phi_\tau$, say, in two stages, first over the shot noise fluctuations (this does not affect the factors $\sin\Phi$, $\sin\Phi_\tau$) and then over the phase of the oscillations. This "double average" can be written in the form

$$\langle \zeta_1 \zeta_{1\tau} \rangle = I_0^{-2} \langle \langle I_f I_{f\tau} \rangle \sin\Phi \sin\Phi_\tau \rangle . \tag{6.47}$$

But the inner average over the shot noise fluctuations, which does not affect the phase of the oscillations, is just the average giving the correlation function (6.24), and hence

$$\langle \zeta_1 \zeta_{1\tau} \rangle = \Gamma^2 e I_0^{-2} \delta(\tau) \langle \langle I_a(t) \rangle \sin^2 \Phi \rangle \tag{6.48}$$

(here we neglect any time dependence of the space-charge smoothing factor Γ^2). Substitution of (6.43) into (6.48) gives

$$\begin{aligned} \langle \zeta_1 \zeta_{1\tau} \rangle = {} & \tfrac{1}{2} \Gamma^2 e I_0^{-2} \\ & \times \langle (i_0 + i_1 \cos\Phi' + i_2 \cos 2\Phi' + i_3 \cos 3\Phi')(1 + \cos 2\Phi') \rangle , \end{aligned} \tag{6.49}$$

where

$$\Phi' = \Phi + \frac{\pi}{2}.$$

When we average over the phase, the only surviving average containing trigonometric functions is $\langle \cos^2 2\Phi' \rangle = \frac{1}{2}$, since the initial phase is assumed to be uniformly distributed. Thus (6.49) reduces to

$$\langle \zeta_1 \zeta_{1\tau} \rangle = \tfrac{1}{2} \Gamma^2 e I_0^{-2} \left(i_0 + \frac{i_2}{2} \right) \delta(\tau). \tag{6.50}$$

Similarly, it is easily verified that

$$\begin{aligned} \langle \zeta_2 \zeta_{2\tau} \rangle &= \tfrac{1}{2} \Gamma^2 e I_0^{-2} \left(i_0 - \frac{i_2}{2} \right) \delta(\tau), \\ \langle \zeta_1 \zeta_{2\tau} \rangle &= 0 . \end{aligned} \tag{6.51}$$

In the particular case (6.45), these formulas give

$$\begin{aligned} \langle \zeta_1 \zeta_{1\tau} \rangle &= \tfrac{1}{2} \Gamma^2 e I_0^{-2} (I_= + \tfrac{1}{8} \beta A_0^2) \delta(\tau), \\ \langle \zeta_2 \zeta_{2\tau} \rangle &= \tfrac{1}{2} \Gamma^2 e I_0^{-2} (I_= + \tfrac{1}{8} \beta A_0^2) \delta(\tau). \end{aligned} \tag{6.52}$$

Thus we see that the periodic nonstationarity of the shot noise

affects the amplitude and phase differently. In fact, the intensity of the fluctuations acting on the amplitude is increased while that of the fluctuations acting on the phase is decreased (for $\beta>0$). The rest of the argument is the same as before, and is based on the equations

$$\begin{aligned} \delta\dot{a} + \varepsilon\omega_0\delta a &= \omega_0\zeta_1, \\ \delta\dot{\psi} &= \omega_0\zeta_2. \end{aligned} \tag{6.53}$$

The increased intensity of the fluctuations acting on the amplitude causes the amplitude fluctuations

$$\langle(\delta a)^2\rangle = \frac{\Gamma^2}{4\varepsilon} e\left(i_0 + \frac{i_2}{2}\right)\frac{\omega_0}{I_0^2} \tag{6.54}$$

to be increased $1+(i_2/2i_0)$ times compared with (6.27).[2] At the same time, the phase diffusion becomes less intense, and the diffusion coefficient in (6.32) becomes

$$D_0 = \tfrac{1}{2}\Gamma^2 e\left(i_0 - \frac{i_2}{2}\right)\frac{\omega_0^2}{I_0^2}. \tag{6.55}$$

Correspondingly, (6.54) and (6.55) lead to changes in the spectral density of the generated signal: The width of the discrete spectral line is diminished by $(50i_2/i_0)\%$, but its intensity remains unchanged. On the other hand, the width $\varepsilon\omega_0$ of the broad spectral component due to the amplitude fluctuations does not change, but its intensity is increased $(50i_2/i_0)\%$.

In the example considered above, we have

$$A = 11v, \quad I_= = 0.046a, \quad \beta = 1.8\times 10^{-4}a/v^2,$$

and hence, according to (6.45), the changes just described amount to about 5%. Then the effect of the fluctuations on the amplitude will be about 10% greater than their effect on the phase.

The analysis just given shows that the size of the effect produced by the nonstationarity of the shot noise is characteristically determined not by the first harmonic of the oscillations of the anode current but by the second harmonic.

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[2] This can be seen by setting $\langle I_a\rangle = i_0$ in (6.27).

4. Influence of Amplitude Fluctuations on Phase Diffusion

Equation (4.20) describing the operation of a vacuum-tube oscillator within the context of the present idealization leads to the equations (6.8) for the first approximation, according to which the amplitude fluctuations do not affect the phase at all. The effect of the amplitude fluctuations on the phase enters only in the second approximation (6.15) and is therefore small. However, in some cases the amplitude fluctuations can have an appreciable effect on the phase fluctuations, for reasons not taken into consideration in the derivation of (4.20), e.g., because of the presence of grid currents. Hence it is of interest to investigate this phenomenon further. We shall do so by starting from the following linearized equation for the phase deviation:

$$\delta\dot{\varphi} = -\frac{\varepsilon\omega_0}{A_0}\gamma\,\delta A - \frac{\varepsilon\omega_0}{A_0}\zeta(t). \tag{6.56}$$

Here $\zeta(t)=\xi(t)\sin(\omega_1 t+\varphi_0)$ is the random process figuring in (4.67) or (6.16), which after "simplification" has the correlation function

$$\langle\zeta\zeta_\tau\rangle = \frac{\kappa(\omega_0)}{2}\delta(\tau)$$

[cf. (4.68)]. The coefficient γ describes the extent to which the amplitude influences the phase, and equals

$$\gamma = \frac{\varepsilon}{4}(1+\tfrac{1}{3}\varrho^2) \tag{6.57}$$

in the case where (6.15) holds. In the linearized approximation, the amplitude fluctuations constitute a Gaussian random process, which is independent of $\zeta(t)$ and has the correlation function (5.38). If we neglect the difference between ω_1 and ω_0, (5.38) takes the form

$$\langle\delta A\,\delta A_\tau\rangle = \sigma^2 e^{-\varepsilon\omega_0|\tau|}, \tag{6.58}$$

where

$$\sigma^2 = \tfrac{1}{4}\varepsilon\omega_0\kappa(\omega_0).$$

Integrating (6.56), we obtain the following expression for the phase

increment after the time T:

$$\Delta\varphi = -\frac{\varepsilon\omega_0}{A_0}\gamma\int_0^T \delta A\, dt - \frac{\varepsilon\omega_0}{A_0}\int_0^T \zeta\, dt. \tag{6.59}$$

Then, taking account of (6.58), we find the expression

$$\begin{aligned}\langle(\Delta\varphi)^2\rangle &= \left(\frac{\varepsilon\omega_0}{A_0}\right)^2\left[\gamma^2\int_0^T\int_0^T \langle\delta A(t)\,\delta A(t')\rangle\, dt\, dt'\right.\\ &\quad\left.+\int_0^T\int_0^T \langle\zeta(t)\zeta(t')\rangle\, dt\, dt'\right]\\ &= \frac{1}{2}\left(\frac{\varepsilon\omega_0}{A_0}\right)^2\kappa(\omega_0)\left[\gamma^2\left(T - \frac{1-e^{-\varepsilon\omega_0 T}}{\varepsilon\omega_0}\right) + T\right]\end{aligned} \tag{6.60}$$

for the variance $\langle(\Delta\varphi)^2\rangle$, while the cross correlation $\langle\Delta A\Delta\varphi\rangle$ is

$$\begin{aligned}&\langle\delta A\Delta\varphi\rangle = \langle\delta A_T\Delta\varphi\rangle = k(T),\\ &k(T) = \frac{\varepsilon\omega_0}{A_0}\gamma\int_0^T \langle\delta A\,\delta A_\tau\rangle\, d\tau = -\frac{\sigma^2}{A_0}\gamma(1-e^{-\varepsilon\omega_0 T}).\end{aligned} \tag{6.61}$$

As the time increases until $T \gg 1/\varepsilon\omega_0$, formula (6.60) gives a purely diffusional law for the phase growth, i.e.,

$$\langle(\Delta\varphi)^2\rangle = D_1 T,$$

where

$$D_1 = D_0(1+2\gamma^2) \qquad \left[D_0 = \frac{1}{2}\left(\frac{\varepsilon\omega_0}{A_0}\right)^2\kappa(\omega_0)\right] \tag{6.62}$$

is the corrected diffusion coefficient, which takes account of the effect of the amplitude fluctuations. At the same time, the correlation (6.61) approaches the limiting value

$$\langle\delta A\,\delta\varphi\rangle = \langle\delta A_T\,\delta\varphi\rangle = -\frac{\sigma^2\gamma}{A_0}. \tag{6.63}$$

The correlation between the amplitude and the phase leads to a

certain modification of the correlation function

$$x = \mathrm{Re}(A e^{i\omega_0 t + i\varphi}) = \mathrm{Re}(z e^{i\omega_0 t})$$

of the generated signal. In the present case, using formulas (1.7.16) and (1.7.73) instead of (1.7.76), we have

$$\langle x x_\tau \rangle = \tfrac{1}{2}\mathrm{Re}(\langle A A_\tau e^{-i\Delta\varphi}\rangle e^{-i\omega_0\tau})$$

or

$$\langle x x_\tau \rangle = \tfrac{1}{2}\mathrm{Re}\{\langle (A_0^2 + A_0\delta A + A_0\delta A_\tau + \delta A\,\delta A_\tau) e^{-i\Delta\varphi}\rangle e^{-i\omega_0\tau}\}. \quad (6.64)$$

The mean value on the right can be calculated from the formula

$$\langle (A_0\,\delta A + A_0\,\delta A_\tau + \delta A\,\delta A_\tau) e^{-i\Delta\varphi}\rangle = \left[\left(A_0\frac{\partial}{i\,\partial u} + A_0\frac{\partial}{i\,\partial v} - \frac{\partial^2}{\partial u\,\partial v}\right)\Theta(u,v,w)\right]_{\substack{u=v=0\\ w=-1}}, \quad (6.65)$$

in terms of the characteristic function

$$\Theta(u,v,w) = \langle e^{iu\,\delta A + iv\,\delta A_\tau + iw\Delta\varphi}\rangle,$$

which, according to (1.31), has the form

$$\Theta(u,v,w) = \exp\{-\tfrac{1}{2}[u^2\langle(\delta A)^2\rangle + 2uv\langle\delta A\,\delta A_\tau\rangle + v^2\langle(\delta A_\tau)^2\rangle + 2uwk(\tau) + 2vwk(\tau) + w^2\langle(\Delta\varphi)^2\rangle\}. \quad (6.66)$$

Substitution of (6.66) into (6.65) gives

$$\langle (A_0 + \delta A)(A_0 + \delta A_\tau) e^{-i\Delta\varphi}\rangle = [A_0^2 - 2iA_0k(\tau) - k^2(\tau) + \langle\delta A\,\delta A_\tau\rangle] e^{-\langle(\Delta\varphi)^2\rangle/2}.$$

It then follows from (6.64) that

$$\langle x x_\tau\rangle = [A_0^2 + \langle\delta A\,\delta A_\tau\rangle - k^2(\tau)] e^{-\langle(\Delta\varphi)^2\rangle/2}\cos\omega_0\tau - 2A_0k(\tau)e^{-\langle(\Delta\varphi)^2\rangle/2}\sin\omega_0\tau. \quad (6.67)$$

Comparing this expression with formula (I.7.4), we define the functions

$$r(\tau) = [A_0^2 + \langle\delta A\,\delta A_\tau\rangle - k^2(\tau)] e^{-D\varphi/2} \approx [A_0^2 + \langle\delta A\,\delta A_\tau\rangle] e^{-D\varphi/2},$$
$$s(\tau) = 2A_0k(\tau)e^{-D\varphi/2}, \quad (6.68)$$

whose Fourier transforms then determine the spectral density $S[x;\omega]$

of the generated signal, as shown by formula (1.7.8). The nonzero odd function $s(\tau)$ now introduces an asymmetry into the form of the spectral density. Using (1.7.6), (6.61) and (6.68) and the smallness of the variance $\mathbf{D}\varphi$ for $\tau \sim \varepsilon\omega_0$, we find that

$$\begin{aligned}\sigma(\Omega) &= -2\mathrm{Im}\int_0^\infty e^{-i\Omega t}s(t)\,dt \\ &\approx 4\sigma^2\gamma\mathrm{Im}\int_0^\infty e^{-i\Omega t}\left(e^{-D_1 t/2}-e^{-\varepsilon\omega_0 t}\right)dt \qquad (6.69)\\ &= -4\sigma^2\gamma\left(\frac{\Omega}{\Omega^2+\frac{1}{4}D_1^2}-\frac{\Omega}{\Omega^2+\varepsilon^2\omega_0^2}\right).\end{aligned}$$

Thus, according to (1.7.8), the effect of the amplitude fluctuations on the phase is to add the asymmetric function

$$\begin{aligned}-4\sigma^2\gamma(\omega-\omega_0)&\left[\frac{1}{(\omega-\omega_0)^2+\frac{1}{4}D_1^2}-\frac{1}{(\omega-\omega_0)^2+\varepsilon^2\omega_0^2}\right]\\ &\approx -4\sigma^2\gamma\frac{\omega-\omega_0}{(\omega-\omega_0)^2+\frac{1}{4}D_1^2}\cdot\frac{\varepsilon^2\omega_0^2}{(\omega-\omega_0)^2+\varepsilon^2\omega_0^2}\end{aligned}$$

to the spectral density of the generated signal. In other words, the symmetry of the spectral line of the oscillation is destroyed by the presence of correlation between the amplitude and the phase.

CHAPTER 7

Effect of Strong External Noise on an Oscillator

We now examine the operation of a vacuum-tube oscillator under the action of fluctuating external voltages of high intensity. In practice, such voltages are sometimes due to unwanted noise, but in some cases they are deliberately applied to the oscillator for specific purposes, for example, to obtain oscillations modulated by noise. In Chap. 4 we derived equation (4.13) describing the operation of a vacuum-tube oscillator when the external voltages are applied 1) to the grid or the anode, and 2) to the inductive branch of the tank circuit (see Figure 14). It is clear from (4.13) that in the first case, the external voltage appears in the argument of the nonlinear function F, while in the second case, the external voltage appears as a separate additive term. This difference leads to corresponding differences both in the mathematical treatment of the problem and in the analytical results obtained. Thus, in what follows, we shall consider each case separately.

1. Phase Fluctuations due to Noise Applied to the Inductive Branch

We begin by examining the case where the external noise is applied to the inductive branch. Neglecting the reaction of the anode load (i.e., setting $D=0$), we write (4.13) as

$$\ddot{I}+\omega_0^2 I=\omega_0^2[F(M\dot{I})-RC\dot{I}]+\omega_0^2 C\dot{u}_b. \tag{7.1}$$

Then the variable $x=\omega_0 M(I-I_=)$, which has the dimensions of a voltage and is proportional to the a–c component of the current I, satisfies the equation

$$\ddot{x}+\omega_0^2 x=\omega_0^3\left[MF_0\left(\frac{\dot{x}}{\omega_0}\right)-RC\frac{\dot{x}}{\omega_0}\right]+\omega_0^3 MC\dot{u}_b, \tag{7.2}$$

where

$$F_0(u)=F(u)-I_==Su+\frac{\beta}{2}u^2-\frac{\gamma}{3}u^3,$$

in terms of the usual approximation of the tube characteristic (see Figure 15). Here the term $\beta u^2/2$ does not affect the simplified equations in the first approximation, and hence can be omitted. Writing

$$\varepsilon\xi(t)=\omega_0 MC\dot{u}_b, \tag{7.3}$$

so that

$$\kappa(\omega)=\tfrac{1}{2}\left(\frac{\omega_0 MC}{\varepsilon}\right)^2\omega^2 S[u_b;\omega],$$

we can reduce equation (7.2) to the same form as equation (4.24), already considered in Chaps. 4 and 5.

As shown in Chap. 4, simplification of the equations in standard form determining the amplitude A and phase φ of the generated signal

$$x=A\cos\Phi=aA_0\cos(\omega_0 t+\varphi)\qquad(x=-\omega_0 A\sin\Phi)$$

leads to equation (4.80), or to

$$\begin{gathered}\dot{a}=\frac{\varepsilon\omega_0}{2}\left(a-a^3+\frac{c}{2a}\right)+\eta',\\ \dot{\varphi}=\frac{\zeta'}{a}\end{gathered} \tag{7.4}$$

in the first approximation. Here $\eta'(t)$ and $\zeta'(t)$ are identically distributed independent Gaussian random processes, with zero means and correlation functions

$$\langle\eta'\eta'_\tau\rangle=\langle\zeta'\zeta'_\tau\rangle=\frac{\varepsilon\omega_0}{2}c\delta(\tau), \tag{7.5}$$

where, according to (4.79) and (7.3),

$$c = \frac{\varepsilon\omega_0 \kappa(\omega_0)}{\overline{A_0^2}} = \frac{\omega_0}{2\varepsilon U_0^2} S[u_b; \omega_0] \tag{7.6}$$

($U_0 = LA_0/M$ is the amplitude of the voltage oscillations across the inductance). Here the coefficient c is comparable to 1, since in this chapter we assume that the noise intensity is high.

In Chap. 5, Sec. 1 we considered the amplitude fluctuations described by the first of the equations (7.4). It was found that the stationary amplitude distribution is given by (5.8) and has the form shown in Figure 16. We also calculated the moments $\langle a^k \rangle$ of various orders, starting from $\langle a^{-1} \rangle$. However, the phase fluctuations described by the second of the equations (7.4) were not considered. We now turn our attention to this problem.

To find the size of the random phase shift $\Delta\varphi$ which occurs during the interval from t_0 to $t_0+\tau$, we integrate the equation $\dot{\varphi} = \zeta'/a$, obtaining

$$\Delta\varphi = \int_{t_0}^{t_0+\tau} \frac{\zeta'(t)}{a(t)} dt. \tag{7.7}$$

Here $a(t)$, like $\zeta'(t)$, is a random function, but is not delta-correlated. It is important to note that $a(t)$ is statistically independent of $\zeta'(t)$, being completely determined by the independent random function $\zeta'(t)$ [see the first of the equations (7.4)]. We are interested in calculating the probability density $w(\Delta\varphi)$ of the phase shift $\Delta\varphi$. Setting $t_0 = 0$ and using formula (1.1.8), we can write $w(\Delta\varphi)$ as an average of a delta function:

$$w(\psi) = \langle \delta(\psi - \Delta\varphi) \rangle = \left\langle \delta\left(\psi - \int_0^\tau \frac{\zeta'(t)}{a(t)} dt\right) \right\rangle. \tag{7.8}$$

The average in (7.8) will be carried out in two stages, by using the fact that $a(t)$ and $\zeta'(t)$ are independent. First, for a fixed realization $a(t)$,

we average over the ensemble of functions $\zeta'(t)$, obtaining the conditional probability density

$$w[\psi \mid a(t)] = \left\langle \delta\left(\psi - \int_0^\tau \frac{\zeta'(t)}{a(t)}\,dt\right)\right\rangle_{\text{fixed } a(t)}. \tag{7.9}$$

Then we regard $a(t)$ as a random function rather than as a fixed function. This makes (7.9) itself random, and to obtain the true probability density (7.8), we must still average (7.9) over all possible realizations of the process $a(t)$:

$$w(\psi) = \langle w(\psi \mid a(t)\mid\rangle. \tag{7.10}$$

Since $\zeta'(t)$ is a Gaussian random function, the integral $\int(\zeta'/a)dt$ is also Gaussian for any fixed $a(t)$. Taking account of the delta-function shape of the correlation function of ζ', we find the conditional variance

$$\langle(\Delta\varphi)^2\rangle_{\text{fixed } a(t)} = \int_0^\tau\int_0^\tau \frac{\langle\zeta'(t)\zeta'(t')\rangle}{a(t)a(t')}\,dt\,dt' = \frac{\varepsilon\omega_0}{2}c\int_0^\tau \frac{dt}{a^2(t)}. \tag{7.11}$$

This variance (together with the zero mean value) uniquely determines the Gaussian probability density

$$w[\psi \mid a(t)] = \pi\varepsilon\omega_0 c\left(\int_0^\tau \frac{dt}{a^2(t)}\right)^{-1/2} \exp\left\{-\frac{\psi^2}{\varepsilon\omega_0 c\int_0^\tau \frac{dt}{a^2(t)}}\right\}. \tag{7.12}$$

and all that remains is to carry out the average over $a(t)$, as described by (7.10).

It should be noted that under stationary conditions, the phase shift of the generated signal has infinite variance. In fact, after averaging the conditional variance (7.11) over $a(t)$, we obtain

$$\langle(\Delta\varphi)^2\rangle = \frac{\varepsilon\omega_0}{2}c\int_0^\tau \langle a^{-2}(t)\rangle\,dt = \frac{\varepsilon\omega_0}{2}c\langle a^{-2}\rangle\tau, \tag{7.13}$$

since $\langle a^{-2}(t)\rangle$ does not depend on t. But it follows from the distribution (5.8) that $\langle a^{-2}\rangle$ is infinite, since the integral

$$a^{-2} = \frac{2}{N}\int_0^\infty e^{-(a^2-1)^2/2c}\frac{da}{a}$$

diverges at the lower limit $a=0$. This particularly strong influence of the small amplitude fluctuations on the phase difference has the following explanation: An appreciable diminution in the amplitude of the oscillations occurs relatively often, i.e., the self-excited oscillations stop almost completely and then start up again, but these inherently "new" oscillations have a phase which is only very weakly related to that of the "old" oscillations. Naturally, this lack of "phase continuity" (which becomes more pronounced as the amplitude falls to lower levels) enhances the amount of phase diffusion. If measures are taken to prevent the amplitude from falling to the zero level, then the phase shift will have finite variance. Note that instead of using the variance, we can characterize the scatter of $\Delta\varphi$ by the average absolute value $\langle|\Delta\varphi|\rangle$, which has the finite value

$$\langle|\Delta\varphi|\rangle = \left(\frac{\varepsilon\omega_0 c}{\pi}\right)^{1/2}\left\langle\left(\int_0^\tau \frac{dt}{a^2(t)}\right)^{1/2}\right\rangle. \tag{7.14}$$

Since $a(t)$ is a complicated random process, it is a difficult matter to carry out calculations based on formulas (7.12), (7.10) and (7.14) in their general form. Therefore we confine ourselves to the case of small time intervals τ, i.e., time intervals during which the value of the amplitude does not manage to change very much. Although greater accuracy is actually possible, we shall use the very simple approximation

$$\int_0^\tau \frac{dt}{a^2(t)} \approx \frac{\tau}{a^2(0)}, \tag{7.15}$$

obtained by assuming that the amplitude remains constant during the interval τ. Substitution of (7.15) into (7.12), (7.10) and (7.14) gives the

formulas

$$w(\psi) = (\pi c\varepsilon\omega_0\tau)^{-1/2}\left\langle a \exp\left\{-\frac{\psi^2 a^2}{c\varepsilon\omega_0\tau}\right\}\right\rangle, \tag{7.16}$$

$$\langle|\Delta\varphi|\rangle = \left(\frac{c\varepsilon\omega_0\tau}{\pi}\right)^{1/2}\langle a^{-1}\rangle, \tag{7.17}$$

involving averages of the instantaneous values of the amplitude. To evaluate the expressions (7.16) and (7.17), we need only know the probability density (5.8). Thus for $w(\psi)$ we obtain the expression

$$w(\psi) = \frac{2}{N\sqrt{\pi c\varepsilon\omega_0\tau}} e^{-1/2c}\int_0^\infty \exp\left\{-\frac{a^4}{2c} + \frac{a^2}{c} - \frac{\psi^2 a^2}{c\varepsilon\omega_0\tau}\right\} a\,da. \tag{7.18}$$

which can be written in the form

$$w(\psi) = \frac{1}{N_2} Z_2\left(\frac{\frac{\psi^2}{\varepsilon\omega_0\tau} - 1}{\sqrt{8c}}\right), \tag{7.19}$$

in terms of the normalization constant

$$N_2 = N(8c)^{-1/4}(\pi\varepsilon\omega_0\tau)^{1/2} e^{1/2c}$$

and the function $Z_2(y)$ introduced in formula (5.16). The function (5.16) can be expressed in terms of cylinder functions by using (5.18). Then the asymptotic formulas for the cylinder functions imply that

$$Z_2(y) \approx \frac{\sqrt{\pi}}{16} y^{-3/2}$$

for large values of y. It follows that

$$w(\psi) \approx \frac{1}{N_2}\frac{\sqrt{\pi}}{16}(8c)^{3/4}(\varepsilon\omega_0\tau)^{3/2}(\psi^2 - \varepsilon\omega_0\tau)^{-3/2} \sim \text{const}\,\psi^{-3}. \tag{7.20}$$

Thus we see that the probability density $w(\psi)$ falls off slowly as ψ increases, so that the integral

$$\langle(\Delta\psi)^2\rangle = \int \psi^2 w(\psi)\,d\psi$$

actually diverges, unlike the integral

$$\langle|\psi|\rangle = \int |\psi|\, w(\psi)\, d\psi\,.$$

In Figure 19 we plot the curves corresponding to the function (7.19) for the values $c=\frac{1}{8}$, $\frac{1}{2}$ and 2. It is apparent from the above considerations that for high intensity noise, the process of phase diffusion is

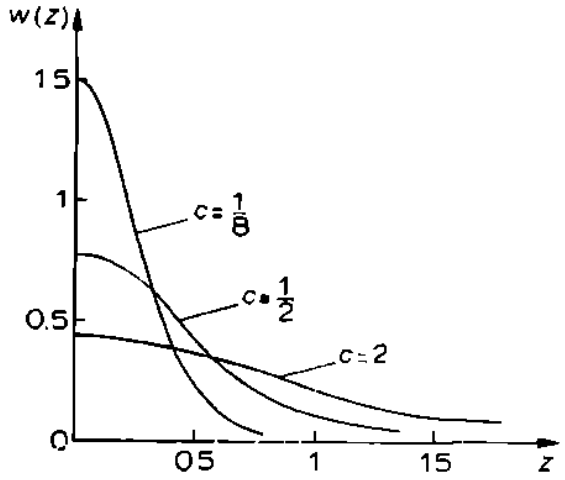

Fig. 19. Probability density of the phase shift $\Delta\varphi$ for various noise levels ($z = \Delta\varphi/\sqrt{\varepsilon\omega_0\tau}$).

drastically different from an ordinary diffusion process, since the diffusion coefficient is formally infinite ($D=\infty$) and the distribution law (7.19) is non-Gaussian.

A problem of great interest is to calculate the correlation function and spectral density of the signal

$$x = A\cos(\omega_0 t + \varphi) = A_0\,\mathrm{Re}\,(ae^{i\varphi + i\omega_0 t})\,.$$

According to (1.7.16), the correlation function can be written as

$$\langle xx_\tau\rangle = \frac{A_0^2}{2}\,\mathrm{Re}\,\{\langle aa_\tau e^{-i\Delta\varphi}\rangle\, e^{-i\omega_0\tau}\}\,.$$

Substituting (7.7) into this expression, we proceed as before, averaging over the statistical ensemble of functions ζ'. After using (7.11), this

gives

$$\langle xx_\tau\rangle = \frac{A_0^2}{2}\left\langle aa_\tau \exp\left\{-\frac{1}{4}\varepsilon\omega_0 c\int_t^{t+\tau}\frac{dt}{a^2(t)}\right\}\right\rangle \cos\omega_0\tau. \quad (7.21)$$

For small τ, when a_τ differs only slightly from a, allowing us to use (7.15), we can write (7.21) in the form

$$\langle xx_\tau\rangle \approx \frac{A_0^2}{2}\langle a^2 e^{-\varepsilon\omega_0 c|\tau|/4a^2}\rangle \cos\omega_0\tau, \quad (7.22)$$

where the sign $\langle\cdots\rangle$ denotes averaging with respect to the weight (5.8). Expanding $\exp\{-\omega_0 c|\tau|/4a^2\}$ in a Taylor series, we find the size of the "corner" of the correlation function (7.21) at the origin:

$$\langle xx_\tau\rangle \approx \frac{A_0^2}{2}\left[\langle a^2\rangle - \frac{\varepsilon\omega_0 c}{4}|\tau| + R_1\right]\cos\omega_0\tau. \quad (7.23)$$

The remainder, denoted by R_1, no longer has corners, i.e., its derivative with respect to τ is continuous. The size of the corner in the correlation function determines the spectral components which fall off most slowly as the frequency increases, allowing us to derive an asymptotic formula for the spectral density $S[x;\omega]$.

When the noise is weak ($c \ll 1$), we can use the linear approximation, setting $a = a_\tau = 1$ in (7.21). This gives

$$\langle xx_\tau\rangle = \frac{A_0^2}{4} e^{-\varepsilon\omega_0 c|\tau|/4}\cos\omega_0\tau, \quad (7.24)$$

which resembles the main contribution to the correlation function (6.40) found earlier. In this approximation, an ordinary diffusion process takes place, with diffusion coefficient

$$D = \frac{\varepsilon\omega_0}{2}$$

[see (5.43)], which determines the width of the spectral line generated by the signal, since the correlation function (7.24) corresponds to the spectral density

$$S[x;\omega] = \frac{A_0^2}{4}\frac{\frac{\varepsilon\omega_0 c}{4}}{(\omega-\omega_0)^2 + (\frac{1}{4}\varepsilon\omega_0 c)^2}.$$

For $|\omega-\omega_0| \gg \varepsilon\omega_0 c$ we have the asymptotic formula

$$S[x;\omega] \approx \frac{A_0^2}{4} \frac{\varepsilon\omega_0 c}{(\omega-\omega_0)^2}, \tag{7.25}$$

corresponding to the corner of the function (7.24) at the origin:

$$\langle x x_\tau \rangle = \frac{A_0^2}{2}\left[1 - \frac{\varepsilon\omega_0 c}{4}|\tau| + R_2\right]\cos\omega_0\tau.$$

But, according to (7.23), the function (7.21) has a corner (i.e., a discontinuity in the derivative) of exactly the same size. It follows that (6.25) is valid for $|\omega-\omega_0| \gg \varepsilon\omega_0 c$ even when the noise intensity is high ($c \sim 1$), despite the fact that the phase diffusion coefficient is infinite. According to formula (I.2.24), the spectral density corresponding to (7.22) is

$$S[x;\omega] = \frac{A_0^2}{4}\left\langle \frac{\varepsilon\omega_0 c a^2}{a^2(\omega-\omega_0)^2 + \left(\frac{1}{4a}\varepsilon\omega_0 c\right)^2}\right\rangle,$$

which, as asserted, goes into the asymptotic expression (7.25) for frequencies $|\omega-\omega_0| \gg \varepsilon\omega_0 c$.

We conclude our discussion of the case where the noise is applied to the inductive branch of the tank circuit, by calculating the value of c for the example considered in Chap. 4. Sec. 2. Substituting

$$U_0 = \frac{11 \times 0.16}{0.18} \approx 100v, \quad \varepsilon \approx 0.05$$

into (7.6), we obtain

$$c \approx \frac{\omega_0}{1000v^2} S[u_b;\omega].$$

Thus, in calculating c, instead of the mean square value of the fluctuating voltage u_b, one should think in terms of a conventional "equivalent" quantity, defined as

$$\sigma_{eq} = (\omega_0 S[u_b;\omega])^{1/2}. \tag{7.26}$$

In our example, c is comparable to 1 if (7.26) is of order $30v$.

2. Amplitude Fluctuations due to Noise Applied to the Grid Circuit

In the case where the external voltage fluctuations act on the grid of the vacuum-tube oscillator, the random function is subjected to a non-linear transformation. In fact, setting $u_b=0$, $D=0$, $u_g=u$ in (4.14) and (4.15), we obtain

$$\ddot{I}+\omega_0^2 I=\omega_0^2\{F[M\dot{I}+u(t)]-RC\dot{I}\}. \tag{7.27}$$

Writing $\dot{x}=\omega_0 M(I-I_=)$ as before, and using our previous approximation (4.22) for the tube characteristic, we find that the oscillations are described by the equation

$$\begin{aligned}\ddot{x}+\omega_0^2 x=\omega_0^3 MS\left(\frac{\dot{x}}{\omega_0}+u\right)+\omega_0^3 M\frac{\beta}{2}\left(\frac{\dot{x}}{\omega_0}+u\right)^2\\ -\omega_0^3 M\frac{\gamma}{3}\left(\frac{\dot{x}}{\omega_0}+u\right)^3-\omega_0^2 RC\dot{x}.\end{aligned} \tag{7.28}$$

Although all the terms in the right-hand side of (7.28) must be small, we use the following notation instead of introducing a small parameter ε:

$$\begin{aligned}q_0&=\omega_0 M\left(Su+\frac{\beta}{2}u^2-\frac{\gamma}{3}u^3\right),\\ q_1&=\omega_0(MS-RC)+\omega_0 M(\beta u-\gamma u^2),\\ q_2&=\omega_0 M\left(\frac{\beta}{2}-\gamma u\right),\\ q_3&=-\omega_0 M\frac{\gamma}{3}.\end{aligned} \tag{7.29}$$

Then (7.28) takes the form

$$\ddot{x}+\omega_0^2 x=\omega_0^2\left[q_0+q_1\frac{\dot{x}}{\omega_0}+q_2\left(\frac{\dot{x}}{\omega_0}\right)^2+q_3\left(\frac{\dot{x}}{\omega_0}\right)^3\right]. \tag{7.30}$$

The equations in standard form (4.31) corresponding to (7.29) are

$$\begin{aligned}\dot{A}&=\frac{\omega_0}{A}\left[q_0\frac{\dot{x}}{\omega_0}+q_1\left(\frac{\dot{x}}{\omega_0}\right)^2+q_2\left(\frac{\dot{x}}{\omega_0}\right)^3+q_3\left(\frac{\dot{x}}{\omega_0}\right)^4\right],\\ \dot{\varphi}&=-\frac{\omega_0}{A^2}\left[q_0+q_1\frac{\dot{x}}{\omega_0}+q_2\left(\frac{\dot{x}}{\omega_0}\right)^2+q_3\left(\frac{\dot{x}}{\omega_0}\right)^3\right]x,\end{aligned} \tag{7.31}$$

where

$$x = A\cos\Phi, \quad \frac{\dot{x}}{\omega_0} = -A\sin\Phi \quad (\Phi = \omega_0 t + \varphi).$$

It is convenient to go from powers and products of trigonometric functions to trigonometric functions of multiple angles. It is easily verified that

$$\begin{aligned}\dot{A} &= \frac{\omega_0}{A}\left[\tfrac{1}{2}q_1A^2 + \tfrac{3}{8}q_3A^4 - q_0A\sin\Phi - \tfrac{3}{4}q_2A^3\sin\Phi\right.\\ &\quad \left. - \tfrac{1}{2}q_1A^2\cos 2\Phi - \tfrac{1}{2}q_3A^4\cos 2\Phi + \ldots\right],\\ \dot{\varphi} &= -\frac{\omega_0}{A^2}\left[q_0A\cos\Phi + \tfrac{1}{4}q_2A^3\cos\Phi - \tfrac{1}{2}q_1A^2\sin 2\Phi\right.\\ &\quad \left. - \tfrac{1}{4}q_3A^4\sin 2\Phi + \ldots\right],\end{aligned} \tag{7.32}$$

where the dots denote similar terms containing sines and cosines of 3Φ and 4Φ.

The fact that the random excitation $u(t)$ undergoes a nonlinear transformation leads to the appearance of a large number of fluctuational terms in the standardized equations (7.32). This is to be contrasted with the earlier situation, where the standardized equations (4.36) contained the random function $\xi(t)$ only in the combinations $\xi\sin\Phi$ and $\xi\cos\Phi$, involving the sine and cosine of the fundamental frequency, so that only the spectral density $\kappa(\omega_0)$ of $\xi(t)$ at the fundamental frequency had an effect in the first approximation. Now the excitation $u(t)$ appears in (7.32) in various combinations involving the trigonometric functions of frequency $0, \omega_0, 2\omega_0, 3\omega_0$ and $4\omega_0$. For example, the expression $\frac{1}{2}q_1A^2$ contains the term $\frac{1}{2}\omega_0 M\beta u A^2$ involving no trigonometric function, the expression $q_0A\sin\Phi$ contains (among others) a term $\omega_0 MSu\sin\Phi$ involving the sine of the fundamental frequency, the term $\frac{1}{2}q_1A^2\cos 2\Phi$ contains the product of βu and a harmonic of twice the fundamental frequency, and so on. Therefore, to a first approximation, the generated signal is appreciably affected by the spectral density $S[u;\omega]$ of the random excitation $u(t)$ near the zero frequency and also near multiples of the fundamental frequency. The presence of terms of the type $\gamma u^2, \gamma u^3\sin\Phi$, etc, containing powers of the random function u, causes the spectral density at other frequencies to have an effect as well.

Of course, the situation becomes simpler if the random excitation $u(t)$ has a small bandwidth compared to the frequency ω_0. Then certain terms in (7.32) have only a slight effect, and the number of important terms becomes smaller. Thus we shall consider the case where the spectral density of the excitation $u(t)$ is appreciably different from zero only in a comparatively narrow band $\Delta\omega \ll \omega_0$ lying in the region of small frequencies. This means that $u(t)$ is slowly varying compared to the high-frequency oscillations, and hence changes appreciably only during a time τ_{cor} which is much larger than a period:

$$\tau_{cor} \gg \frac{2\pi}{\omega_0}.$$

If we average the right-hand sides of (7.32) over a period, the terms with trigonometric functions drop out, since the coefficients multiplying them (which depend on A and u) change only slightly during a period. In the first approximation, this gives the truncated equations

$$\dot{A} = \frac{\omega_0}{2} A\left(q_1 + \tfrac{3}{4} q_3 A^2\right), \quad \dot{\varphi} = 0.$$

We now write the expression (7.29) for q_1 as a sum of two terms

$$q_1 = 2\mu + 2\zeta(t), \tag{7.33}$$

where

$$2\mu = \omega_0 (MS - RC) + \omega_0 M \langle \beta u - \gamma u^2 \rangle, \tag{7.34}$$

and $\zeta(t)$ is the random function

$$\zeta = \frac{\omega_0 M}{2} [\beta u - \gamma u^2 - \langle \beta u - \gamma u^2 \rangle], \tag{7.35}$$

with mean zero and correlation function

$$\begin{aligned} \langle \zeta \zeta_\tau \rangle &= \left(\frac{\omega_0 M}{2}\right)^2 \mathbf{K}[\beta u - \gamma u^2, \beta u_\tau - \gamma u_\tau^2] \\ &= \left(\frac{\omega_0 M}{2}\right)^2 \{\beta^2 \mathbf{K}[u, u_\tau] - \beta\gamma \mathbf{K}[u, u_\tau^2] \\ &\quad - \beta\gamma \mathbf{K}[u^2, u_\tau] + \gamma^2 \mathbf{K}[u^2, u_\tau^2]\}. \end{aligned} \tag{7.36}$$

Assuming that u is a stationary Gaussian process with correlation function

$$\langle uu_\tau \rangle \equiv k(\tau) = \sigma^2 R(\tau)$$

and mean zero, we have

$$\mathbf{K}[u, u_\tau] = k(\tau), \quad \mathbf{K}[u, u_\tau^2] = 0.$$

To calculate the expression

$$\mathbf{K}[u^2, u_\tau^2] \equiv \langle u^2 u_\tau^2 \rangle - \langle u^2 \rangle^2,$$

we use formula (I.1.70), with $t_1 = t_2 = t$, $t_3 = t_4 = t + \tau$ and $k_4 = 0$, obtaining

$$\langle u^2 u_\tau^2 \rangle = 2k^2(\tau) + k^2(0) = 2\sigma^4 R^2(\tau) + \sigma^4.$$

Therefore the correlation function (7.36) becomes

$$\langle \zeta \zeta_\tau \rangle = \left(\frac{\omega_0 M}{2}\right)^2 \sigma^2 [\beta^2 R(\tau) + 2\gamma^2 \sigma^2 R^2(\tau)]. \tag{7.37}$$

With the notation (7.34) and (7.35), the equation describing the amplitude fluctuations takes the form

$$\dot{A} = \omega_0 A(\mu + \tfrac{3}{8} q_3 A^2) + \omega_0 A \zeta(t). \tag{7.38}$$

If the correlation time τ_{cor} of the externally applied noise is less than the amplitude relaxation time, so that

$$\tau_{cor} \ll \frac{1}{\mu \omega_0},$$

then (7.38) can be solved by using the Fokker-Planck equation. Here we represent the random function $\zeta(t)$ as a delta-correlated process, replacing the correlation function (7.37) by

$$\langle \zeta \zeta_\tau \rangle = K\delta(\tau),$$

with

$$K = \int_{-\infty}^{\infty} \langle \zeta \zeta_\tau \rangle \, d\tau = \left(\frac{\omega_0 M \sigma}{2}\right)^2 \left[\beta^2 \int_{-\infty}^{\infty} R(\tau)\, d\tau + 2\gamma^2 \sigma^2 \int_{-\infty}^{\infty} R^2(\tau)\, d\tau\right]. \tag{7.39}$$

By the usual rules (see Chap. I.4), the Fokker-Planck equation

corresponding to (7.38) is

$$w(A) = -\frac{\partial}{\partial A}\left[\omega_0 A(\mu + \tfrac{3}{8}q_3A^2)w\right] + \tfrac{1}{4}K\omega_0^2\frac{\partial}{\partial A}\left[\frac{\partial}{\partial A}(A^2w) + A^2\frac{\partial w}{\partial A}\right] \tag{7.40}$$

[cf. (I.4.161), (I.4.162) and (I.4.170)]. The stationary amplitude distribution is characterized by vanishing probability current:

$$\omega_0 A(\mu + \tfrac{3}{8}q_3A^2)w - \tfrac{1}{4}K\omega_0^2\left[\frac{\partial}{\partial A}(A^2w) + A^2\frac{\partial w}{\partial A}\right] = 0. \tag{7.41}$$

Introducing the notation

$$\Lambda = -\tfrac{3}{8}\frac{q_3}{\omega_0 K}, \quad \nu = \frac{\mu}{\omega_0 K}, \tag{7.42}$$

and integrating (7.41), we obtain the stationary distribution

$$w(A) = \frac{2\Lambda^\nu}{\Gamma(\nu)}A^{2\nu-1}e^{-\Lambda A^2} \tag{7.43}$$

plotted in Figure 20 [cf. (I.4.49) and (I.4.171)].

It is apparent from (7.43) that the quantity Λ determines the scale along the A-axis, in the sense that the amplitude appears in the

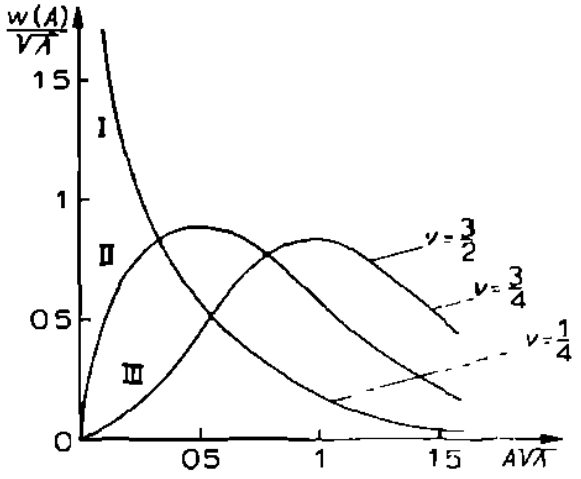

Fig. 20. Probability density of the amplitude for noise applied to the grid circuit: I) Undeveloped oscillations; II) Partially developed oscillations; III) Fully developed oscillations.

expression for the element of probability dP in the combination $A\sqrt{\Lambda} \equiv B$:

$$dP = w(A)\, dA = \frac{2}{\Gamma(\nu)}\, e^{-B^2} B^{2\nu-1}\, dB.$$

However, the shape of the curve is determined by the quantity ν, the ratio between μ and an "equivalent noise variance" $\omega_0 K$. In fact, using the form of the amplitude distribution curve, we can make the following qualitative characterization of the operating regime of the oscillator:

1. If $\mu < 0$ ($\nu < 0$), there are no oscillations at all, at the point representing the value of the amplitude simply stays at the origin of coordinates, regardless of the size of the noise fluctuations.
2. For small positive values of μ such that

$$0 < \mu < \tfrac{1}{2}\omega_0 K, \quad \text{i.e.,} \quad 0 < \nu < \tfrac{1}{2},$$

the power $2\nu - 1$ of A in (7.43) is negative. However, despite the resulting infinite value of the probability density of the amplitude at the origin, the density remains integrable, i.e., the probability of occurrence of small amplitudes is small. This is the case where the external noise is large compared to the "strength" of the limit cycle. In fact, the noise is so large that the self-excited oscillations are repeatedly thrown out of the stable cycle, preventing the generated oscillations from having any "predominant" amplitude. Then, as can be seen from Figure 20, the probability density is monotonic, i.e., the smaller the amplitude, the larger its probability. This might be called the regime of "undeveloped oscillations."
3. When the relative noise level becomes smaller, so that

$$\tfrac{1}{2}\omega_0 K < \mu < \omega_0 K, \quad \text{i.e.,} \quad \tfrac{1}{2} < \nu < 1,$$

the probability density is no longer a monotonic curve. In fact, in this case, there is a most probable value of the amplitude, equal to

$$A_m = \sqrt{\frac{2\nu - 1}{2\Lambda}}, \tag{7.44}$$

but this maximum is not very pronounced (see Figure 20). In fact,

the amplitude of the oscillations is scattered over a wide bandwidth and can take values near zero with appreciable probability. This might be called the regime of "partially developed oscillations."

4. Finally let $\mu > \omega_0 K$ ($\nu > 1$), corresponding to an even smaller noise level or a more stable limit cycle. Then we have the most interesting case, i.e., the regime of "fully developed oscillations", where the representative point stays predominantly near the limit cycle. Then the scatter about the most probable value (7.44) plays a subordinate role, and becomes smaller and smaller as ν is increased.

Using (7.43), we can easily calculate the moments of the amplitude distribution, obtaining

$$\langle A^k \rangle = \frac{2\Lambda^\nu}{\Gamma(\nu)} \int_0^\infty A^{2\nu+k-1} e^{-\Lambda A^2} dA = \frac{\Gamma(\nu + \frac{1}{2}k)}{\Lambda^{k/2}\Gamma(\nu)}. \tag{7.45}$$

The formulas for the even moments are particularly simple:

$$\langle A^2 \rangle = \frac{\nu}{\Lambda}, \langle A^4 \rangle = \frac{(\nu+1)\nu}{\Lambda^2}, \ldots, \langle A^{2l} \rangle = \frac{(\nu+l-1)\ldots(\nu+1)\nu}{\Lambda^l}. \tag{7.46}$$

The mean value of the amplitude is given by

$$\langle A \rangle = \frac{1}{\sqrt{\Lambda}} \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)}, \tag{7.47}$$

which differs from the expression (7.44) for the most probable value. Together, (7.46) and (7.47) give the variance

$$\mathbf{D}A = \langle A^2 \rangle - \langle A \rangle^2 = \frac{1}{\Lambda}\left[\nu - \frac{\Gamma^2(\nu+\frac{1}{2})}{\Gamma^2(\nu)}\right]. \tag{7.48}$$

Using (7.45), we can immediately write the odd moments in terms of the first moment:

$$\langle A^3 \rangle = \frac{\nu+\frac{1}{2}}{\Lambda}\langle A \rangle, \langle A^5 \rangle = \frac{(\nu+\frac{3}{2})(\nu+\frac{1}{2})}{\Lambda^2}\langle A \rangle \ldots. \tag{7.49}$$

Thus we have expressed everything in terms of elementary functions except the mean value $\langle A \rangle$. However, there is a simple approximate

formula suitable for large values of ν. In fact, it is easily deduced from the well-known asymptotic representation of the gamma function that

$$\langle A\rangle = \sqrt{\frac{\bar{\nu}}{\Lambda}}\, e^{-1/8\nu}\left[1 + O\left(\frac{1}{\nu^2}\right)\right], \tag{7.50}$$

which, together with (7.48), implies

$$\mathbf{D}A = \frac{1}{4\Lambda}\left[1 + O\left(\frac{1}{\nu}\right)\right]. \tag{7.51}$$

We conclude this section by making some numerical estimates, after establishing the relation between the parameters μ, ν and the previous quantities ε, A_0. It follows from (4.25), (7.29) and (7.34) that

$$\mu = \frac{\varepsilon}{2} - \frac{1}{2}\omega_0 M\gamma\sigma^2 = \frac{\varepsilon}{2}\left(1 - 4\frac{\sigma^2}{A_0^2}\right),$$

$$-\frac{3}{8}q_3 = \frac{1}{8}\omega_0 M\gamma = \frac{\varepsilon}{2A_0^2},$$

and hence

$$\Lambda = \frac{\varepsilon}{2A_0^2\omega_0 K}, \quad \nu = \frac{\varepsilon}{2\omega_0 K}\left(1 - 4\frac{\sigma^2}{A_0^2}\right). \tag{7.52}$$

But, according to (7.39),

$$K = \left(2\varepsilon\frac{\sigma^2}{A_0^2}\right)^2\left[\frac{\beta^2}{\gamma^2\sigma^2}\int_{-\infty}^{\infty} R(\tau)\,d\tau + 2\int_{-\infty}^{\infty} R^2(\tau)\,d\tau\right]$$

$$= \frac{\varepsilon^2}{2}\left(\frac{2\sigma}{A_0}\right)^4\left(\frac{\beta^2}{\gamma^2\sigma^2}\tau_{cor} + 2\tau'_{cor}\right),$$

where

$$\tau_{cor} = \int_0^{\infty} R(\tau)\,d\tau, \quad \tau'_{cor} = \int_0^{\infty} R^2(\tau)\,d\tau.$$

Substituting

$$A_0 = 1/v, \quad \frac{\beta}{\gamma} = 3v,$$

we obtain

$$2\omega_0 K = \varepsilon^2 \omega_0 \tau_{cor} \left(\frac{\sigma}{5.5}\right)^4 \left[\left(\frac{3}{\sigma}\right)^2 + 1\right], \tag{7.53}$$

if $\tau'_{cor} \approx \tau_{cor}$. For example, if $\varepsilon = 0.1$, $\omega_0 \tau_{cor} = 10$, then

$$\nu \approx 10 \left(\frac{5.5}{\sigma}\right)^2 \frac{\left(\frac{5.5}{\sigma}\right)^2 - 1}{\left(\frac{3}{\sigma}\right)^2 + 1}.$$

It follows that the regime of fully developed oscillations is upset ($\nu = 1$) when the noise is increased much past the value $\sigma = 5v$. For values of $\sigma = 5.5v$ and higher, the oscillations are completely extinguished.

3. The Amplitude Correlation Function and the Spectral Density

Having found the stationary amplitude distribution, we now study the time variations of the amplitude. Introducing the dimensionless time

$$t_0 = \omega_0^2 K t, \tag{7.54}$$

we can write equation (7.38) describing the amplitude variations in the particularly simple form

$$\frac{dA}{dt_0} = \nu A - \Lambda A^3 + A\zeta_0(t_0), \tag{7.55}$$

where $\zeta_0 = \zeta/\omega_0 K$ is a random process with correlation function

$$\langle \zeta_0(t'_0)\zeta_0(t_0)\rangle = \delta(t'_0 - t_0). \tag{7.56}$$

Our problem is to find the amplitude correlation function $\langle A(t'_0)A(t_0)\rangle$, or the quantity $v_2(t_0 - t'_0)$, in terms of the notation

$$v_k(t_0 - t'_0) = \langle A(t'_0)A^{k-1}(t_0)\rangle \quad (k = 1, 2, \ldots). \tag{7.57}$$

We begin by differentiating $\langle A(t'_0)A(t_0)\rangle$ with respect to t_0. After

using (7.55), this gives

$$\left\langle A' \frac{dA}{dt_0} \right\rangle = v\langle A'A\rangle - \Lambda\langle A'A^3\rangle + \langle A'A\zeta_0\rangle, \qquad (7.58)$$

where $A' = A(t_0')$. To calculate the last term, we must take account of the statistical dependence between $A(t_0)$ and $\zeta_0(t_0)$. Because of the small correlation time of the process ζ_0, the first term $A_{-\Delta}$ of the sum

$$A(t_0) = A(t_0 - \Delta) + \Delta A \equiv A_{-\Delta} + \Delta A$$

does not depend on $\zeta_0(t_0)$ even for small $\Delta \ll 1/v$, provided that $\Delta \gg \tau_{cor}$, and hence

$$\langle A'A_{-\Delta}\zeta_0\rangle = \langle A'A_{-\Delta}\rangle\langle\zeta_0\rangle.$$

But this term vanishes, since ζ_0 has mean zero, and therefore

$$\langle A'A\zeta_0\rangle = \langle A'A_{-\Delta}\zeta_0\rangle + \langle A'\Delta A\zeta_0\rangle = \langle A'\Delta A\zeta_0\rangle.$$

Substituting into this equation the expression

$$\Delta A = (vA_{-\Delta} - \Lambda A_{-\Delta}^3)\Delta + A_{-\Delta}\int_{t_0-\Delta}^{t_0} \zeta_0\, dt_0'$$

implied by (7.55), we obtain

$$\langle A'A\zeta_0\rangle = \left\langle A'A_{-\Delta}\int_{t_0-\Delta}^{t_0} \zeta_0(t_0')\, dt_0' \cdot \zeta_0(t_0)\right\rangle.$$

Again using the independence of $A_{-\Delta}$ and $\zeta_0(t_0)$, we replace the average on the right by a product of averages:

$$\langle A'A\zeta_0\rangle = \langle A'A_{-\Delta}\rangle\int_{t_0-\Delta}^{t_0} \langle\zeta_0(t_0)\zeta_0(t_0')\rangle\, dt_0'.$$

Since Δ is much larger than the correlation time, the integral on the right is just half the intensity coefficient, i.e.,

$$\int_{t_0-\Delta}^{t_0} \langle\zeta_0(t_0)\zeta_0(t_0')\rangle\, dt_0' = \int_{-\infty}^{0} \langle\zeta_0(t_0)\zeta_0(t+\tau)\rangle\, d\tau = \tfrac{1}{2},$$

and hence

$$\langle A'A\zeta_0\rangle = \tfrac{1}{2}\langle A'A\rangle. \tag{7.59}$$

In writing (7.59), we have also used the fact that the difference $A - A_{-\Delta}$ is small, because of the inequality $\Delta \ll 1/\nu$. In the same way, we can easily prove the more general relation

$$\langle A(t_0')f[A(t_0)]\zeta_0(t_0)\rangle \equiv \langle A'f(A)\zeta_0\rangle = \frac{1}{2}\left\langle A(t_0')\frac{\partial f[A(t_0)]}{\partial A}A(t_0)\right\rangle. \tag{7.60}$$

which is approximate for a finite but small correlation time and exact for the delta-correlated process ζ_0.

Substituting (7.59) into (7.58), we obtain

$$\frac{dv_2(\tau)}{d\tau} = (\nu + \tfrac{1}{2})v_2(\tau) - \Lambda v_4(\tau). \tag{7.61}$$

in the notation of (7.57). Similarly, because of (7.55),

$$\left\langle A'\frac{dA^3}{dt_0}\right\rangle = 3\nu\langle A'A^3\rangle - 3\Lambda\langle A'A^5\rangle + 3\langle A'A^3\zeta_0\rangle.$$

or

$$\frac{dv_4(\tau)}{d\tau} = 3(\nu + \tfrac{1}{2})v_4(\tau) - 3\Lambda v_6(\tau). \tag{7.62}$$

after using (7.60). Continuation of this process involves functions (7.57) of progressively higher orders, which in turn satisfy suitable equations like (7.61) and (7.62). As a result, we obtain an infinite chain of differential equations determining the functions (7.57), with the obvious initial conditions

$$v_2(0) = \langle A^2\rangle, \quad v_4(0) = \langle A^4\rangle, \ldots, \tag{7.63}$$

where the univariate moments $\langle A^2\rangle$, $\langle A^4\rangle$, ... have already been found [see (7.45)]. This chain of equations can be solved by various approximation methods. One such method consists in truncating the chain, by considering only a finite number of functions $v_2, v_4, \ldots, v_{2l}$, and somehow expressing the function v_{2l+2} appearing in the equation for v_{2l} in terms of functions with lower indices. It is best to express v_{2l+2} in terms of the function v_{2l} whose behavior most closely resembles that of v_{2l+2}.

The following method of representing v_{2l+2} in terms of v_{2l} gives good results: As the time difference $t_0 - t_0'$ is increased ($t_0 - t_0' \to \infty$), each function v_k turns into a product of moments:

$$v_k \to \langle A \rangle \langle A^{k-1} \rangle .$$

This suggests introducing the correlation coefficient

$$\frac{v_k - \langle A \rangle \langle A^{k-1} \rangle}{\langle A^k \rangle - \langle A \rangle \langle A^{k-1} \rangle} \quad (k = 1, 2, \ldots) \tag{7.64}$$

of $A(t_0')$ and $A^{k-1}(t_0)$, normalized in such a way as to equal unity when $t_0 = t_0'$. The coefficients (7.64) are in general different for different values of k. However, it is intuitively clear that the way (7.64) depends on time cannot be very different for neighboring values of k, e.g., for $k = 2l$ and $k = 2l+2$. Thus, equating the coefficients (7.64) for $k = 2l$ and $k = 2l+2$, we arrive at one method of expressing v_{2l+2} in terms of v_{2l}. After truncating the chain of equations, the remaining system of equations (which are in fact linear) can be solved in the usual manner. Of course, the more equations are retained, the greater will be the accuracy of the resulting calculation of $v_2(\tau)$.

We now use this method in its simplest version, retaining only one equation (7.61) and expressing $v_4(\tau)$ in terms of $v_2(\tau)$. Equating correlation coefficients

$$\frac{v_4(\tau) - \langle A \rangle \langle A^3 \rangle}{\langle A^4 \rangle - \langle A \rangle \langle A^3 \rangle} = \frac{v_2(\tau) - \langle A \rangle^2}{\langle A^2 \rangle - \langle A \rangle^2},$$

and using the previous formulas for the moments

$$\langle A^3 \rangle = \frac{\nu + \frac{1}{2}}{\Lambda} \langle A \rangle, \quad \langle A^4 \rangle = \frac{\nu + 1}{\Lambda} \langle A^2 \rangle,$$

we obtain

$$v_4(\tau) = \frac{(\nu + 1)\langle A^2 \rangle - (\nu + \frac{1}{2})\langle A \rangle^2}{\Lambda \, \mathbf{D}A} v_2(\tau) - \frac{\langle A^2 \rangle \langle A \rangle^2}{2\Lambda \, \mathbf{D}A}. \tag{7.65}$$

With this approximation, (7.61) goes into the equation

$$\frac{dv_2(\tau)}{d\tau} = -\langle A^2 \rangle \frac{v_2(\tau) - \langle A \rangle^2}{2 \, \mathbf{D}A}. \tag{7.66}$$

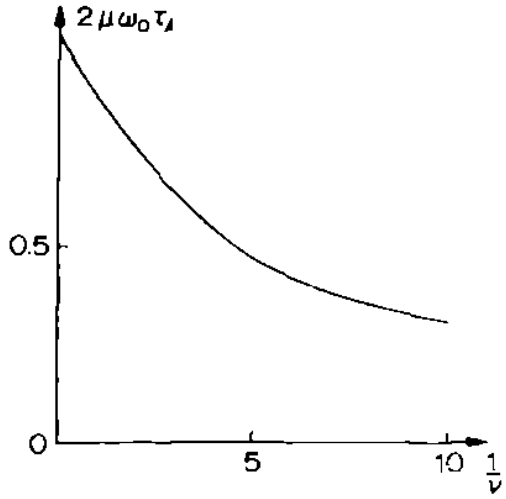

Fig. 21. Dependence of the amplitude correlation time τ_A on the noise intensity.

The solution of (7.66), subject to the initial condition (7.63), is

$$v_2(\tau) = \langle A\rangle^2 + \mathbf{D}A \exp\left\{-\frac{\langle A^2\rangle}{2\,\mathbf{D}A}\tau\right\},$$

or

$$\langle AA_\tau\rangle = \langle A^2\rangle e^{-\tau/\tau_A} + \langle A\rangle^2(1 - e^{-\tau/\tau_A}), \tag{7.67}$$

where we return to ordinary time and introduce an *amplitude correlation time* defined by

$$\tau_A = \frac{1}{\omega_0^2 K}\frac{2\,\mathbf{D}A}{A^2} = \frac{2A}{\omega_0\mu}\mathbf{D}A = \frac{2}{\omega_0\mu}\left[\nu - \frac{\Gamma^2(\nu+\frac{1}{2})}{\Gamma^2(\nu)}\right]. \tag{7.68}$$

Figure 21 shows the dependence of the amplitude correlation time on the noise intensity divided by the "strength" of the limit cycle, i.e., on the quantity

$$\frac{1}{\nu} = \frac{\omega_0 K}{\mu}.$$

The very fact that the correlation time depends on the noise intensity is a typically nonlinear effect. As can be seen from the figure, an increase in the noise intensity leads to a decrease in the correlation time. For small intensities $\nu \gg 1$, it follows from (7.68) and (7.51) that

$$\tau_A \approx \frac{1}{2\mu\omega_0}, \tag{7.69}$$

a result which can also be obtained by the linearization method.

Equation (7.67) shows that in the present approximation, the amplitude fluctuations constitute an exponentially correlated random process with correlation function $\mathbf{D}A \cdot e^{-|\tau|/\tau_A}$, corresponding to the spectral density

$$S[A - \langle A\rangle; \omega] = \frac{4\tau_A \mathbf{D}A}{1 + \omega^2\tau_A^2}. \tag{7.70}$$

It is also easy to write an expression for the spectral density of the generated signal $x = A\cos\omega_0 t$ (in the case under consideration, the effect of phase fluctuations is negligible). After averaging over the random initial phase, we have

$$\langle xx_\tau\rangle = \tfrac{1}{2}\langle AA_\tau\rangle \cos\omega_0\tau, \tag{7.71}$$

or, because of (7.67),

$$\langle xx_\tau\rangle = \tfrac{1}{2}\left[\langle A\rangle^2 + \mathbf{D}A \cdot e^{-|\tau|/\tau_A}\right]\cos\omega_0\tau. \tag{7.72}$$

Since $\tau_A \gg 2\pi/\omega_0$, the corresponding spectral density is

$$S[x;\omega] \approx \pi\langle A^2\rangle\,\delta(\omega - \omega_0) + \frac{\tau_A \mathbf{D}A}{1 + (\omega - \omega_0)^2\tau_A^2} \tag{7.73}$$

[cf. (I.2.23) and (I.2.24)].

A characteristic feature of the present problem, in which relatively slow noise fluctuations are introduced into the grid circuit (where they are subjected to a nonlinear transformation) is that the amplitude fluctuations greatly exceed the phase fluctuations. This accounts for the presence of the sharp line in the spectral density (7.73), in addition to the band due to the amplitude fluctuations.

CHAPTER 8

Effect of Slowly Varying Ambient Noise on an Oscillator

As we saw in Chap. 6, the shot noise and thermal noise invariably present in every oscillator circuit cause only slight fluctuations in the amplitude and phase of the generated signal. However, there are other factors impeding the operating of an oscillator which produce much greater frequency changes, and these must be taken into account in any study of oscillator instability. Such factors include changes in the circuit parameters due to various causes, like temperature fluctuations and mechanical shock. For example, temperature changes can produce corresponding changes in the capacitance and inductance of the tank circuit, or in the cathode emission. Moreover, lack of constancy of the supply voltage may play an important role, there may be further instability in the cathode emission due to the flicker effect, and so on. The totality of all such spurious factors, which produce larger amplitude and phase fluctuations than the "internal noise" discussed in Chap. 7, will be called "ambient noise." Ambient noise is characteristically more slowly varying than internal noise, and it is because of this qualitative difference that internal noise is not completely masked by the stronger ambient noise.

The slowly varying character of ambient noise allows us to apply the quasi-static method of Chap. 5, Sec. 3. Let $\xi(t)$ be the random function describing the fluctuations of some circuit parameter. Then the operation of the oscillator is described by the equations

$$\dot{A} = \varepsilon G(A, \xi), \quad \dot{\varphi} = \varepsilon H(A, \xi) \tag{8.1}$$

[cf. (4.33) and (5.46)]. It will always be assumed that the correlation time of the fluctuations is much greater than the relaxation time, i.e.,

$$\tau_{cor} \gg \frac{1}{\varepsilon\omega_0}. \tag{8.2}$$

Therefore, for every value of $\xi(t)$, a corresponding "quasi-stable" value of the amplitude manages to be established. This value

$$A(t) = A[\xi(t)] \tag{8.3}$$

can be determined from the equation

$$G[A(t), \xi(t)] = 0.$$

Thus the dependence of $A(t)$ on $\xi(t)$ reduces to a zero-memory transformation. Moreover, according to the second of the equations (8.1), the frequency is also a zero-memory transformation of $\xi(t)$:

$$\dot{\varphi} = \Omega(\xi), \quad \Omega(\xi) = \varepsilon H[A(\xi), \xi]. \tag{8.4}$$

The parameter instability permitted in practical oscillator circuits must never produce more than a small amount of amplitude scatter. Therefore the condition

$$\langle(\delta A)^2\rangle \ll A_0^2 \tag{8.5}$$

is always an underlying practical requirement, justifying the use of the linearization method of Chap. 5, Sec. 2. This makes the problem of studying ambient noise even simpler.

Let σ_Ω denote the standard deviation of the frequency fluctuations. Then, before starting our detailed study of ambient noise, it is worth pointing out that a key role will be played by whether the quantity $\sigma_\Omega\tau_{cor}$ is large or small compared to unity.

In this chapter, we shall consider only two kinds of ambient noise, i.e., flicker noise and fluctuations in anode voltage. Although these may not be the most important causes of oscillator instability in every case, they do exhibit all the characteristic features of ambient noise in general. Accordingly, the methods used below, together with the accompanying qualitative conclusions, are for the most part still valid when dealing with other kinds of ambient noise.

1. Self-Excited Oscillations in the Presence of Fluctuations of Anode Voltage

If the anode voltage is unstable, we can imagine that a variable voltage $u_a(t)$ is introduced into the anode circuit, as shown in Figure 14. Setting all the other voltages equal to zero in (4.14) and (4.15), we obtain

$$\ddot{I} + \omega_0^2 I = \omega_0^2 \{F[M'\dot{I} + u(t)] - RC\dot{I}\}. \tag{8.6}$$

This equation is the same as (7.27) with M and $u = u_c$ replaced by $M' = M - DL$ and $u_a = Du_a$, but, of course, D can no longer be set equal to zero, since we want to examine the fluctuations of anode voltage, using formula (4.6) to approximate the characteristic (4.5). It follows from (8.6) that the oscillations of the coordinate $x = \omega_0 M'(I - I_-)$ are described by the equation

$$\ddot{x} + \omega_0^2 x = \omega_0^2 \left[q_0 + q_1 \frac{\dot{x}}{\omega_0} + q_2 \left(\frac{\dot{x}}{\omega_0} \right)^2 + q_3 \left(\frac{\dot{x}}{\omega_0} \right)^3 \right]. \tag{8.7}$$

which coincides with (7.30). Here the coefficients q_i are given by the previous formulas (7.29) with M replaced by M'. The equations in standard form corresponding to (8.7) are given by (7.31), or more concisely by

$$\begin{aligned} \dot{A} &= \frac{\varepsilon}{\omega_0 A} f(y, u)\, y = \frac{\omega_0}{A}(q_0 y + q_1 y^2 + q_2 y^3 + q_3 y^4), \\ \dot{\varphi} &= -\frac{\varepsilon}{\omega_0 A^2} f(y, u)\, x = -\frac{\omega_0}{A^2}(q_0 + q_1 y + q_2 y^2 + q_3 y^3)\, x, \end{aligned} \tag{8.8}$$

where $y = \dot{x}/\omega_0$. Using (8.7), we can write the second of the equations (8.8) in the form

$$\dot{\varphi} = \frac{\varepsilon}{\omega_0 A^2} f \frac{\dot{x} - \varepsilon f}{\omega_0^2} = \frac{\varepsilon}{\omega_0^2 A^2} f(y, u)\dot{y} - \frac{\varepsilon^2}{\omega_0^3 A^2} f^2(y, u). \tag{8.9}$$

If A and u were constant, the first term in the right-hand side of (8.9) would be the time derivative of the quantity

$$\frac{\varepsilon^2}{\omega_0^2 A^2} \tilde{f}(y, u),$$

where

$$\frac{\partial \tilde{f}(y,u)}{\partial y} = f(y,u), \tag{8.10}$$

i.e.,

$$\varepsilon \tilde{f}(y,u) = \omega_0^2 (q_0 y + \tfrac{1}{2} q_1 y^2 + \tfrac{1}{3} q_2 y^3 + \tfrac{1}{4} q_3 y^4). \tag{8.11}$$

Because of the slow variation of the functions $A(t)$ and $u(t)$, this term does not equal the exact time derivative, but involves additional small corrections:

$$\frac{\varepsilon}{\omega_0^2 A^2} f(y,u)\dot{y} = \frac{d}{dt}\left[\frac{\varepsilon}{\omega_0^2 A^2}\tilde{f}(y,u)\right] - \frac{\varepsilon}{\omega_0^2 A^2}\frac{\partial \tilde{f}}{\partial u}\dot{u} + \frac{2\varepsilon}{\omega_0^2 A^3}\tilde{f}\dot{A}.$$

Substituting this expression into (8.9), and using the first of the equations (8.8), we obtain

$$\frac{d}{dt}\left(\varphi - \frac{\varepsilon}{\omega_0^2 A^2}\tilde{f}\right) = -\frac{\varepsilon}{\omega_0^2 A^2}\frac{\partial \tilde{f}}{\partial u}\dot{u} + \frac{\varepsilon^2}{\omega_0^3 A^2} f\left(\frac{2\tilde{f}y}{A^2} - f\right). \tag{8.12}$$

Together, equation (8.12) and the first of the equations (8.8) form an exact system, containing only one rapidly varying function $y = -A\sin(\omega_0 t + \varphi)$. Using the fact that A and u change only slightly during a period, we can average over a period. Then the odd powers of the rapidly varying function y vanish, since they contain only oscillatory components, while even powers of y must be replaced by their nonoscillatory components:

$$y^2 \to \tfrac{1}{2}A^2, \quad y^4 \to \tfrac{3}{8}A^4, \quad y^6 \to \tfrac{5}{16}A^6, \quad y^8 \to \tfrac{35}{128}A^8. \tag{8.13}$$

Thus the first of the equations (8.8) gives

$$\dot{A} = \omega_0 A(\tfrac{1}{2}q_1 + \tfrac{3}{8}q_3 A^2). \tag{8.14}$$

To transform (8.12), we begin with the terms of order ε. Because of (8.11), we have

$$\varepsilon\tilde{f} \to \omega_0^2\left(\tfrac{1}{4}q_1 A^2 + \tfrac{3}{32}q_3 A^4\right), \tag{8.15}$$

where the expression on the right will be denoted simply by $\omega_0^2 A^2 \chi$. Then, using (8.13) to transform the expression $f(\tilde{f}_{\nu A^{-2}} - f)$, we obtain

$$\dot{\varphi} - \dot{\chi} = -\frac{\partial \chi}{\partial u}\dot{u} - \omega_0\left(\tfrac{1}{8}q_1^2 + \tfrac{1}{6}q_2^2 A^2 + \tfrac{9}{32}q_1 q_3 A^2 + \tfrac{45}{256}q_3^2 A^4\right). \tag{8.16}$$

Together, equations (8.14) and (8.16) are an example of the system (8.1), and are valid if $\tau_{cor} \gg 1/\omega_0$. If the condition (8.2) holds, corresponding to even slower fluctuations, then we can go over to the quasi-static approximation. The appropriate equations are obtained by setting $u=0$ and $A=0$ in (8.14) and (8.16). Then (8.14) gives

$$A^2 = -\frac{4}{3}\frac{q_1}{q_3}, \tag{8.17}$$

and substituting (8.17) into (8.16), we obtain

$$\dot{\varphi} = -\omega_0 q_1\left(\frac{1}{16}q_1 - \frac{2}{9}\frac{q_2^2}{q_3}\right). \tag{8.18}$$

To transform (8.18) further, we introduce the notation

$$\begin{gathered} q_1 = 2\mu + 2\zeta, \\ \mu = \frac{\omega_0}{2}(M'S - RC), \quad \dot{\zeta} = \frac{\omega_0 M'}{2}(\beta u - \gamma u^2), \end{gathered} \tag{8.19}$$

which closely resembles that of formulas (7.33) to (7.35), except that we do not center ζ about its mean value. The random function ζ has the same correlation function in both (7.35) and (8.19). If we substitute into (8.18) both (8.19) and the relations

$$\begin{aligned} q_2^2 &= \left(\frac{\omega_0 M'\beta}{2}\right)^2 - 2\omega_0 M'\gamma\zeta \\ q_3 &= -\omega_0 M'\frac{\gamma}{3} \end{aligned}$$

[see (7.29)], we find that

$$\dot{\varphi} = -\omega_0(\mu+\zeta)\left(\frac{\mu+\zeta}{4} + \frac{1}{3\gamma}\omega_0 M'\beta^2 - \tfrac{8}{3}\zeta\right) \tag{8.20}$$
$$= -\frac{\omega_0}{4}(\mu+\zeta)(\mu_1 - \tfrac{29}{3}\zeta),$$

where

$$\mu_1 = \mu + \frac{4}{3\gamma}\omega_0 M'\beta^2.$$

At the same time, the expression (8.17) for the amplitude becomes

$$A^2 = 8\frac{\mu+\zeta}{\omega_0 M'\gamma}. \tag{8.21}$$

It is clear from (8.21) that in the present case, the extra condition (8.5) that the amplitude scatter be small takes the form

$$\langle\zeta^2\rangle \ll \mu^2. \tag{8.22}$$

If (8.22) holds, then the right-hand side of (8.20) can be simplified by dropping quadratic terms, i.e.,

$$\dot{\varphi} = \frac{\omega_0}{4}(\tfrac{29}{3}\mu - \mu_1)\zeta \equiv \omega_0\mu_2\zeta, \tag{8.23}$$

where

$$\mu_2 = \tfrac{26}{12}\mu - \frac{1}{3\gamma}\omega_0 M'\beta^2.$$

Here we also drop the constant term $-\frac{1}{4}\omega_0\mu\mu_1$, which has no effect on the size of the fluctuations but merely represents a correction to the fundamental frequency of the oscillations.

Equation (8.20) and the less exact equation (8.23) are concrete examples of (8.4), which when integrated gives the phase shift after time τ:

$$\Delta\varphi = \int_{t_0}^{t_0+\tau} \Omega dt. \tag{8.24}$$

It is particularly easy to study the behavior of the phase if Ω is a normal random process.

2. The Correlation Function and the Spectral Density of the Signal for Gaussian Frequency Fluctuations

Suppose the fluctuations of anode voltage u_a (and hence of $u = Du_a$ as well) are described by a Gaussian random process, with zero mean and correlation function

$$\langle uu_\tau \rangle = \sigma^2 R(\tau). \tag{8.25}$$

Moreover, suppose the condition (8.22) holds, for which the following two conditions are certainly sufficient:

$$\left(\frac{\omega_0 M'}{2}\right)^2 \beta^2 \sigma^2 \ll \mu^2, \quad \left(\frac{\omega_0 M'}{2}\right)^2 \gamma^2 \sigma^4 \ll \mu^2$$
$$\left(\mu = \frac{\omega_0}{2}(M'S - RC)\right). \tag{8.26}$$

Usually, the order of RC is less than that of $M'S$, so that $M'S - RC \sim M'S$. After dropping RC, the first of the conditions (8.26) takes the form

$$\beta^2 \sigma^2 \ll S^2,$$

or

$$\beta\sigma \ll S. \tag{8.27}$$

Sometimes the inequality

$$\gamma\sigma \ll \beta \tag{8.28}$$

also holds. This means that the quadratic component $-\frac{1}{2}\omega_0 M'\gamma u^2$ of the process ζ is much smaller than the linear component, which allows us to neglect the former and write

$$\zeta = \frac{\omega_0 M'}{2}\beta u. \tag{8.29}$$

But then the frequency Ω appearing in (8.4) and (8.24) is a Gaussian random process with correlation function

$$\langle \Omega\Omega_\tau \rangle = \sigma_\Omega^2 R(\tau) \quad (\sigma_\Omega = \tfrac{1}{2}\mu_2 \omega_0^2 M'\beta) \tag{8.30}$$

and mean zero. In this case, the various statistical characteristics of

the amplitude and phase fluctuations can easily be calculated. In fact, according to (8.22) and (8.23),

$$A = A_0 + \frac{A_0}{2\mu}\zeta, \tag{8.31}$$

and hence the amplitude, as well as the phase, is a Gaussian process.

Having made these preliminary remarks, we now find the correlation function and spectral density of the generated signal $A\cos(\omega_0 t+\varphi)$, assuming that $\dot{\varphi}=\Omega(t)$ is a stationary Gaussian random process. According to (8.5), the amplitude fluctuations make a negligible contribution to the broadening of the discrete spectral line, and hence

$$x = A_0 \cos(\omega_0 t + \varphi),$$
$$\langle x x_\tau \rangle = \tfrac{1}{2}A_0^2 \left\langle \cos\left(\omega_0 \tau + \int_{t_0}^{t_0+\tau} \Omega\, dt\right)\right\rangle, \tag{8.32}$$

where we have used (8.24), averaging over the random initial phase. The correlation function (8.32) can be expressed in terms of the characteristic function of the phase shift:

$$\langle x x_\tau \rangle = \tfrac{1}{2}A_0^2 \operatorname{Re}\left\{e^{i\omega_0\tau}\left\langle \exp\left(i\int_{t_0}^{t_0+\tau} dt\right)\right\rangle\right\}. \tag{8.33}$$

In fact, to write the characteristic function, we need only know the variance of the phase shift, which, according to (8.24) and (8.30) equals

$$\langle(\Delta\varphi)^2\rangle = \sigma_\Omega^2 \int_{t_0}^{t_0+\tau}\int_{t_0}^{t_0+\tau} R(t-t')\,dt\,dt' = 2\sigma_\Omega^2 \int_0^\tau (\tau-\tau')R(\tau')\,d\tau'. \tag{8.34}$$

Therefore

$$\langle x x_\tau \rangle = \tfrac{1}{2}A_0^2 \exp\left\{-\sigma_\Omega^2 \int_0^\tau (\tau-\tau')R(\tau')\,d\tau'\right\}\cos\omega_0\tau, \tag{8.35}$$

and the spectral density $S[x;\omega]$ is then found by taking the Fourier transform of (8.35).

Example 1. Suppose the correlation coefficient $R(\tau)$ is of the form

$$R(\tau) = \tau_{cor}^3 (\tau^2 + \tau_{cor}^2)^{-3/2}. \tag{8.36}$$

Then

$$\langle (\Delta\varphi)^2 \rangle = 2\sigma_\Omega^2 \int_0^\tau (\tau - \tau') R(\tau')\, d\tau' = 2\sigma_\Omega^2 \tau_{cor} (\sqrt{\tau^2 + \tau_{cor}^2} - \tau_{cor}),$$

and hence

$$\langle x x_\tau \rangle = \tfrac{1}{2} A_0^2 \exp\{\sigma_\Omega^2 \tau_{cor}^2 - \sigma_\Omega^2 \tau_{cor} \sqrt{\tau^2 + \tau_{cor}^2}\} \cos \omega_0 \tau\,. \tag{8.37}$$

Using the formula

$$\int_{-\infty}^{\infty} e^{-i\omega z - p\sqrt{z^2+q^2}}\, dz = \frac{2qp}{\sqrt{p^2+\omega^2}} K_1(q\sqrt{p^2+\omega^2}). \tag{8.38}$$

we find the Fourier transform (I.7.6) of the expression

$$r(\tau) = \tfrac{1}{2} A_0^2 \exp\{\sigma_\Omega^2 \tau_{cor}^2 - \sigma_\Omega^2 \tau_{cor} \sqrt{\tau^2 + \tau_{cor}^2} - i\omega_0 \tau\}. \tag{8.39}$$

whose spectral density at positive frequencies, according to (I.7.8), is practically the same as the desired spectral density of (8.37). As a result, we have

$$S[x;\omega] = A_0^2\, e^{\sigma_\Omega^2 \tau_{cor}^2} \frac{\sigma_\Omega^2 \tau_{cor}^2}{\sqrt{\sigma_\Omega^4 \tau_{cor}^2 + (\omega - \omega_0)^2}} \times K_1\left[\tau_{cor} \sqrt{\sigma_\Omega^4 \tau_{cor}^2 + (\omega - \omega_0)^2}\right]. \tag{8.40}$$

Example 2. Suppose the frequency is an exponentially correlated process with correlation function

$$\langle \Omega \Omega_\tau \rangle = \sigma_\Omega^2\, e^{-\beta|\tau|} = A\beta^2\, e^{-\beta|\tau|} \quad \left(\beta = \frac{1}{\tau_{cor}}\right), \tag{8.41}$$

where $\beta = 1/\tau_{cor}$. Then, according to (8.35),

$$\langle x x_\tau \rangle = \tfrac{1}{2} A_0^2 \exp\{-A(\beta\tau - 1 + e^{-\beta\tau})\} \cos \omega_0 \tau \quad (\tau > 0). \tag{8.42}$$

To find the spectral density $S[x;\omega]$, we first calculate the Fourier

transform (1.7.6):

$$\begin{aligned} \varrho(\omega) &= A_0^2 \int_0^\infty \exp\{-\Lambda(\beta\tau - 1 + e^{-\beta\tau})\} \cos \omega\tau \, d\tau \\ &= A_0^2 \operatorname{Re} \int_0^\infty \exp\{-\Lambda(\beta\tau - 1 + e^{-\beta\tau})\} e^{-i\omega\tau} \, d\tau . \end{aligned} \tag{8.43}$$

The change of variable $e^{-\beta\tau} = z$ transforms (8.43) into

$$\varrho(\omega) = \frac{A_0^2}{\beta} e^{\Lambda} \operatorname{Re} \int_0^1 e^{-\Lambda z} z^{\Lambda + \frac{i\omega}{\beta} - 1} \, dz . \tag{8.44}$$

The last integral can be expressed in terms of the incomplete gamma function

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} \, dt$$

after making the substitution $\Lambda z = t$:

$$\begin{aligned} \varrho(\omega) &= \frac{A_0^2}{\beta} e^{\Lambda} \operatorname{Re}\left\{ \Lambda^{-\Lambda - \frac{i\omega}{\beta}} \int_0^\Lambda e^{-t} t^{\Lambda + \frac{i\omega}{\beta} - 1} \, dt \right\} \\ &= \frac{A_0^2}{\beta} e^{\Lambda} \operatorname{Re}\left\{ \Lambda^{-\Lambda - \frac{i\omega}{\beta}} \, \gamma\left(\Lambda + \frac{i\omega}{\beta}, \Lambda\right) \right\}. \end{aligned} \tag{8.45}$$

For practical calculations, it is convenient to replace the exponential $e^{-\Lambda z}$ in (8.44) by its series expansion. This leads to

$$\varrho(\omega) = \frac{A_0^2}{\beta} e^{\Lambda} \operatorname{Re} \sum_{n=0}^\infty \frac{(-\Lambda)^n}{n!} \frac{1}{n + \Lambda + \frac{i\omega}{\beta}},$$

or, after taking the real part,

$$\varrho(\omega) = A_0^2 \beta e^{\Lambda} \sum_{n=0}^\infty \frac{(-\Lambda)^n}{n!} \frac{n + \Lambda}{\omega^2 + (n + \Lambda)^2 \beta^2}. \tag{8.46}$$

Taking account of (8.42), (8.43) and (8.46), and recalling from (I.7.8) that multiplying by the function $\cos \omega_0 t$ in (8.42) leads to a frequency

shift of ω_0 in the calculated spectral density, we obtain

$$S[x;\omega] = \varrho(\omega-\omega_0) = A_0^2\beta e^{A}\sum_{n=0}^{\infty}\frac{(-A)^n}{n!}\cdot\frac{n+A}{(\omega-\omega_0)^2+(n+A)^2\beta^2}. \tag{8.47}$$

Finally we note that, regardless of the form of the correlation coefficient, the following formulas hold for small and large values of τ, respectively:

$$\langle xx_\tau\rangle \approx \tfrac{1}{2}A_0^2\, e^{-\sigma_\Omega^2\tau^2/2}\cos\omega_0\tau \quad \text{for} \quad \tau \ll \tau_{cor}, \tag{8.48}$$

$$\langle xx_\tau\rangle \approx \tfrac{1}{2}A_0^2\, e^{-K_\Omega|\tau|/2}\cos\omega_0\tau \quad \text{for} \quad \tau \gg \tau_{cor}$$

$$\left(K_\Omega = 2\sigma_\Omega^2\int_0^\infty R(\tau)\,d\tau\right). \tag{8.49}$$

3. Effect of Flicker Noise on the Frequency of an Oscillator

We now consider another important special case of Gaussian frequency fluctuations, i.e., the slow fluctuations due to instability in the current emitted by the cathode of the vacuum-tube oscillator. There is reason to believe that modulating fluctuations of anode current, with spectral density

$$S[I_f;\omega] = b\frac{\langle I_a\rangle^2}{\omega^{1+\alpha}} \quad (I_f = I_a - \langle I_a\rangle), \tag{8.50}$$

called "flicker noise," make an important contribution to the frequency instability of self-excited oscillations. The constants b and α in (8.50) are determined from experiment, and $1+\alpha$ usually turns out to be close to unity.

The fluctuations in emission current produce fluctuations in the interelectrode space charge, thereby affecting the size of the interelectrode capacitance. The frequency of the oscillations is given by the formula

$$\omega = \frac{1}{\sqrt{LC_{eq}}}, \tag{8.51}$$

in terms of the equivalent capacitance C_{eq} of the tank circuit. Since C_{eq}

depends on the interelectrode capacitance, and since the latter in turn depends on the anode current I_a, we find that changes in I_a are accompanied by changes in C_{eq}. Suppose this dependence is described by the coefficient μ, defined by the formula

$$\mu = \frac{1}{C_{eq}} \frac{dC_{eq}}{dI_a}, \tag{8.52}$$

where μ can be found experimentally. Assuming that the fluctuations of anode current are relatively small, we can find the corresponding frequency fluctuations by differentiating formula (8.51), i.e.,

$$\dot{\varphi} = \omega - \omega_0 = -\frac{1}{2} \frac{1}{\sqrt{LC_{eq}}} \frac{1}{C_{eq}} \frac{dC_{eq}}{dI_a} I_f,$$

or

$$\dot{\varphi} = \Omega(t) \equiv -\tfrac{1}{2}\mu\omega_0 I_f, \tag{8.53}$$

because of (8.52), where the quantity $\mu\omega_0$ in (8.53) is evaluated for the average value of the emission current. Together, (8.53) and (8.50) imply that the frequency fluctuations have spectral density

$$S[\Omega;\omega] = \frac{a}{\omega^{1+\alpha}}, \tag{8.54}$$

where

$$a = \tfrac{1}{4}\omega_0^2\mu^2\langle I_a\rangle^2.$$

According to formula (I.5.30), if $\alpha=0$, the spectral density (8.54) corresponds to the correlation function

$$\langle\Omega\Omega_\tau\rangle = \frac{a}{2\pi} \ln \frac{\omega_c}{\omega_l^2\tau}. \tag{8.55}$$

Assuming that the fluctuations are Gaussian, we use formula (8.35) to find the correlation function of the generated signal. Direct integration of (8.55) gives

$$\int_0^\tau (\tau - \tau')\langle\Omega\Omega_{\tau'}\rangle\, d\tau' = \frac{a\tau^2}{4\pi} \ln\frac{\gamma}{\tau}, \tag{8.56}$$

where

$$\gamma = \frac{\omega_c}{\omega_l^2} e^{3/2}.$$

Because of the smallness of $1/\omega_c$ [see (8.62) below], there is only a negligible contribution to the integral (8.56) from the region of values $\tau < 1/\omega_c$ where (8.55) does not hold.

Replacing $\sigma_\Omega^2 R(\tau)$ in (8.35) by the expression (8.55) and taking account of (8.56), we find that

$$\langle x x_\tau \rangle = \tfrac{1}{2} A_0^2 \exp\left\{ -\frac{a\tau^2}{4\pi} \ln \frac{\gamma}{\tau} \right\} \cos \omega_0 \tau . \tag{8.57}$$

The exact calculation of the spectral density $S[x; \omega]$ corresponding to (8.57) is difficult, but an approximate expression can be found by using the fact that the logarithm is slowly varying with respect to τ^2. The Fourier transform of (8.57) is appreciably affected only by the region where

$$\frac{a\tau^2}{4\pi} \ln \frac{\gamma}{\tau} \sim 1 ,$$

and in this region the argument τ in the logarithm can be replaced by the particular value τ_0 defined by the formula

$$\frac{a\tau_0^2}{4\pi} \ln \frac{\gamma}{\tau_0} = 1 . \tag{8.58}$$

Then (8.57) takes the form

$$\langle x x_\tau \rangle = \tfrac{1}{2} A_0^2 \exp\left\{ -\frac{a\tau^2}{4\pi} \ln \frac{\gamma}{\tau_0} \right\} \cos \omega_0 \tau ,$$

and hence has the spectral density

$$S[x; \omega] = \frac{\sqrt{\pi}}{\beta} A_0^2 \exp\left\{ -\frac{(\omega - \omega_0)^2}{\beta^2} \right\}, \tag{8.59}$$

where

$$\beta^2 = \frac{a}{\pi} \ln \frac{\gamma}{\tau_0} . \tag{8.60}$$

Equation (8.58) can be written in the form

$$y = \frac{a\gamma^2}{8\pi} \ln y , \tag{8.61}$$

where

$$y = \frac{\gamma^2}{\tau_0^2} .$$

Our previous considerations are based on neglecting the behavior of the correlation function (8.55) for $\tau < 1/\omega_c$ and $\tau > 1/\omega_l$. This is permissible if

$$\frac{a\omega_c^2}{\omega_l^4} \gg 1, \quad \text{i.e.,} \quad \frac{a\gamma^2}{8\pi} \gg 1. \tag{8.62}$$

If (8.62) holds, we can use the formula

$$y_n = \frac{a\gamma^2}{8\pi} \ln y_{n-1}$$

to solve (8.61) by the method of successive approximations. The first approximation is

$$y_1 = \frac{a\gamma^2}{8\pi} = \frac{a\omega_c^2 e^3}{8\pi\omega_l^4} \quad \left(\tau_0 = \sqrt{\frac{8\pi}{a}}\right), \tag{8.63}$$

the second approximation is

$$y_2 = \frac{a\gamma^2}{8\pi} \ln \frac{a\gamma^2}{8\pi},$$

and so on. Substituting (8.63) into (8.60), we find the following expression for the width of the spectral line (8.59):

$$\beta = \sqrt{\frac{a}{2\pi} \ln y} = \sqrt{\frac{a}{2\pi} \ln \frac{a\gamma^2}{8\pi}} \quad \left(\gamma^2 = \frac{\omega_c^2 e^3}{\omega_l^4}\right) \tag{8.64}$$

Formula (8.64) can also be used to study flicker noise as a nonstationary process, provided we replace $1/\omega_l$ by the observation time t, since this is the only difference between formulas (I.5.27) and (I.5.30). As a result, we obtain

$$\beta = \sqrt{\frac{a}{2\pi} \ln \frac{a\omega_c^2 t^4 e^3}{8\pi}}. \tag{8.65}$$

According to (8.65), the width of the spectral line slowly increases with the time of observation. The cutoff frequency ω_c is determined by the smoothing of the fluctuations by various factors, e.g., space-charge and frequency depression effects, parasitic capacitances, etc., and is inversely proportional to the smoothing time, i.e., $\omega_c = 1/\tau_{sm}$.

4. Large Independent Phase Increments

It is possible to regard the frequency $\Omega(t)$ as a Gaussian random process only in certain special cases. Therefore we now study the phase fluctuations by another approach, involving no assumptions about the normality of $\Omega(t)$. According to (8.24), the phase shift $\Delta\varphi$ accumulates as a result of the addition of elementary phase increments $\Omega\, dt$. These increments are statistically dependent, since the correlation time τ_{cor} of the process $\Omega(t)$ has a finite nonzero value. However, choosing intervals somewhat larger than τ_{cor}, we can represent (8.24) as a sum of increments

$$\Delta\varphi = \int_{t_0}^{t_0+\tau_1} \Omega\, dt + \int_{t_0+\tau_1}^{t_0+2\tau_1} \Omega\, dt + \int_{t_0+2\tau_1}^{t_0+3\tau_1} \Omega\, dt + \ldots \quad (\tau_1 > \tau_{cor}). \tag{8.66}$$

which are now independent to a certain approximation. Since the phase appears in the exponential $e^{i\varphi}$, it is important to know the size of the mean value of these increments compared to π, since the statistical picture will depend on this size. In the present section, we consider the case of independent increments which on the average greatly exceed π (the case of small increments will be considered in the next section). This corresponds to the inequality

$$\sigma_\Omega \tau_{cor} \gg 1 \tag{8.67}$$

If τ is the time it takes for a phase shift of order π to accumulate (e.g., $\sigma_\Omega \tau \sim 1$), then (8.67) implies

$$\tau \ll \tau_{cor},$$

so that the frequency Ω does not manage to change very much during the time τ. Therefore, for such a value of τ,

$$\int_{t_0}^{t_0+\tau} \Omega\, dt \approx \Omega\tau \tag{8.68}$$

and

$$\langle e^{i\Delta\varphi} \rangle \approx \int e^{i\Omega\tau} w(\Omega)\, d\Omega. \tag{8.69}$$

Hence according to (8.33),

$$\langle xx_\tau \rangle = \tfrac{1}{2}A_0^2 \operatorname{Re}\left\{e^{i\omega_0\tau}\int e^{i\Omega\tau}w(\Omega)\,d\Omega\right\}. \tag{8.70}$$

If we increase τ, then (8.69) and (8.70) cease to be valid when τ attains values of order τ_{cor}. But before this happens, both the right-hand side of (8.69) and the correlation function $\langle xx_\tau \rangle$ become negligibly small. In fact, it is assumed that the function $w(\Omega)$ satisfies certain continuity requirements and hence has a Fourier transform which falls off rapidly with increasing frequency (increasing τ, in the present case). Therefore (8.69) takes large values in the region $\tau \sim 1/\sigma_\Omega$, but its values become small if $\tau \gg 1/\sigma_\Omega$. To see that $\langle xx_\tau \rangle$ becomes small for $\tau \gg 1/\sigma_\Omega$, we need only consider intermediate values $1/\sigma_\Omega \ll \tau \ll \tau_{cor}$ for which (8.70) continues to hold although the right-hand side has already fallen off sharply.

Since both sides of formula (8.70) are small in the region where it breaks down, we can use (8.70) for all τ when calculating the spectral density. Naturally, the Fourier transform of the characteristic function will involve the original probability density, and in fact, the spectral density corresponding to (8.70) is just

$$S[x;\omega] = \pi A_0^2[w_\Omega(\omega-\omega_0) + w_\Omega(-\omega-\omega_0)]. \tag{8.71}$$

As before, we neglect the effect of the tail of the function $w_\Omega(-\omega-\omega_0)$ entering the region of positive frequencies from the region of negative frequencies (this is justified by the narrow-band character of the signal). Then (8.71) becomes

$$S[x;\omega] = \pi A_0^2 w_\Omega(\omega-\omega_0). \tag{8.72}$$

Thus the spectral density is completely determined by the probability density of the frequency. Although this result seems obvious at first, it is valid only for the case of large independent phase increments, i.e., for the case where the frequency fluctuations either have large variance or are of long duration.

Example 1. If $\Omega(t)$ is a Gaussian random process, then it follows at once from (8.72) that

$$S[x;\omega] = \sqrt{\frac{\pi}{2}}\,\frac{1}{\sigma_\Omega}\,A_0^2\,e^{-(\omega-\omega_0)^2/2\sigma_\Omega^2}, \tag{8.73}$$

which corresponds to (8.48), as might be expected. When the correlation function has the form (8.36), this formula can be obtained from (8.40). In fact, because of (8.67), the argument of the function K_1 is large, allowing us to replace K_1 by its asymptotic expansion, which gives

$$S[x;\omega] = A_0^2\sqrt{\frac{\pi}{2}}\frac{\sigma_\Omega^2\tau_{cor}^3}{[\tau_{cor}\sqrt{\sigma_\Omega^4\tau_{cor}^2+(\omega-\omega_0)^2}]^{3/2}} \times \exp\{\sigma_\Omega^2\tau_{cor}^2 - \tau_{cor}\sqrt{\sigma_\Omega^4\tau_{cor}^2+(\omega-\omega_0)^2}\}.$$

Since

$$\sigma_\Omega^2\tau_{cor}^2\left[1-\sqrt{1+\frac{(\omega-\omega_0)^2}{\sigma_\Omega^4\tau_{cor}^2}}\right] \approx -\frac{(\omega-\omega_0)^2}{2\sigma_\Omega^2},$$

$$\sqrt{\sigma_\Omega^4\tau_{cor}^2+(\omega-\omega_0)^2} \approx \sigma_\Omega^2\tau_{cor}$$

for $\omega-\omega_0 \ll \sigma_\Omega^2\tau_{cor}$, the spectral density $S[x;\omega]$ immediately reduces to (8.73) for $|\omega-\omega_0| \ll \sigma_\Omega^2\tau_{cor}$.

Example 2. Next suppose $\Omega(t)$, instead of being a Gaussian process, is proportional to the square of a Gaussian process. This will be the case if $\beta=0$, since then

$$\zeta = -\tfrac{1}{2}\omega_0 M'\gamma u^2 = -4\mu\frac{u^2}{A_0^2},$$

according to (8.19), where

$$A_0^2 = \frac{8\mu}{\omega_0 M'\gamma}$$

[cf. (8.21)]. Substitution of ζ into (8.23) gives

$$\Omega = -\tfrac{26}{3}\omega_0\mu^2\frac{u^2}{A_0^2}, \tag{8.74}$$

where u has the probability density

$$w(u) = \frac{1}{\sqrt{2\pi}\sigma}e^{-u^2/2\sigma^2}.$$

Solving (8.74) for u and substituting the result into the expression for $w(u)$, we obtain

$$w(\Omega) = w(u)\left|\frac{du}{d\Omega}\right| = \frac{A_0}{4\mu\sigma}\left(\frac{3}{13\pi\omega_0|\Omega|}\right)^{1/2} \times \exp\left\{-\frac{3A_0^2}{4\mu^2\sigma^2}\frac{|\Omega|}{13\omega_0}\right\}, \quad (\Omega<0).$$

Therefore (8.72) implies

$$S[x;\omega] = \frac{A_0^3}{4\mu\sigma}\left[\frac{3\pi}{13\omega_0(\omega_0-\omega)}\right]^{1/2} \exp\left\{-\frac{3A_0^2}{4\mu^2\sigma^2}\frac{\omega_0-\omega}{13\omega_0}\right\} \quad \text{for} \quad \omega < \omega_0, \tag{8.75}$$

$$S[x;\omega] = 0 \quad \text{for} \quad \omega > \omega_0.$$

There is no essential difficulty in finding $S[x;\omega]$ in the more general case where $\beta \neq 0$, or in the case where (8.20) has to be used instead of (8.23), because of the large size of $\langle \zeta^2 \rangle$.

5. Small Independent Phase Increments

Next we consider frequency fluctuations whose correlation time is much less than the time it takes for large phase shifts of order π to accumulate. In this case

$$\sigma_\Omega \tau_{cor} \ll 1, \tag{8.76}$$

so that the large phase shifts are essentially sums of large numbers of independent random variables. Therefore the phase shift which accumulates during times of order

$$\tau \gtrsim \frac{1}{\sigma_\Omega} \gg \tau_{cor} \tag{8.77}$$

has an approximately Gaussian distribution (due to the operation of the central limit theorem), with characteristic function

$$\langle e^{iu\Delta\varphi} \rangle = e^{-u^2\langle(\Delta\varphi)^2\rangle/2}, \tag{8.78}$$

where

$$\langle(\Delta\varphi)^2\rangle = \int_{-\tau}^{\tau} (\tau - |\tau'|)\langle \Omega\Omega_{\tau'}\rangle \, d\tau' \approx K_\Omega \tau$$

$$\left(K_\Omega = 2\int_0^\infty \langle \Omega\Omega_\tau \rangle \, d\tau\right),$$

because of the condition (8.77). It follows from (8.33) and (8.78) that

$$\langle x x_\tau \rangle = \tfrac{1}{2}A_0^2 \, e^{-K_\Omega|\tau|/2} \cos\omega_0\tau \qquad (\tau \gg \tau_{cor}). \tag{8.79}$$

Equation (8.79) fails to hold only in the range $\tau \lesssim \tau_{cor}$, which is narrow compared to the effective width of the function $e^{-K\Omega\tau/2}$, equal to

$$\frac{1}{K_\Omega} \sim \frac{1}{\sigma_\Omega^2 \tau_{cor}}.$$

In calculating the spectral density corresponding to (8.79), the interval $\tau \lesssim \tau_{cor}$ can be neglected for frequencies $\omega \lesssim K$, since it forms only a small part of a period. Therefore, using (8.79) for all τ, we obtain

$$S[x;\omega] = \tfrac{1}{2}A_0^2 \frac{K_\Omega}{(\omega-\omega_0)^2 + \frac{1}{4}K^2} \quad (|\omega - \omega_0| \lesssim K). \quad (8.80)$$

Using (8.80), we can find the broadening of the discrete spectral line due to the fluctuations, since this requires knowledge of the spectral density in precisely the region of small frequency deviations $\omega - \omega_0$. In fact, it is clear from (8.80) that the frequency instability broadens the spectral line by an amount

$$\Delta\omega \approx K_\Omega = \int_{-\infty}^{\infty} \langle \Omega\Omega_\tau \rangle \, d\tau. \quad (8.81)$$

One is often interested in the value of the spectral density for larger deviations (of order $\varepsilon\omega_0$) from the fundamental frequency, i.e., for

$$|\omega - \omega_0| \gtrsim \frac{1}{\tau_{cor}},$$

since the slowness of the fluctuations makes the quantity $1/\tau_{cor}$ rather small ($\ll \varepsilon\omega_0$). Because of (8.76), only small phase shifts accumulate during time intervals of order τ_{cor}, i.e.,

$$\int_{t_0}^{t_0+\tau} \Omega \, dt \ll 1, \quad e^{i\Delta\varphi} \approx 1 + i\Delta\varphi - \tfrac{1}{2}(\Delta\varphi)^2 + \ldots \quad (8.82)$$

Therefore

$$\langle e^{i\Delta\varphi} \rangle \approx 1 - \frac{1}{2} \int_{t_0}^{t_0+\tau} \int_{t_0}^{t_0+\tau} \langle \Omega(t)\Omega(t') \rangle \, dt \, dt' \quad (8.83)$$

for $\tau \ll 1/\sigma_\Omega$, and

$$\langle xx_\tau \rangle = \tfrac{1}{2}A_0^2 \left[1 - \tfrac{1}{2}\int_0^\tau (\tau - \tau')\langle \Omega_t \Omega_{t'} \rangle \, d\tau'\right] \cos \omega_0 \tau. \quad (8.84)$$

To calculate the spectral density for $\omega - \omega_0 \gtrsim 1/\tau_{cor}$, we first find the spectral density of the process

$$\eta(t) = \frac{d}{dt} e^{-i\varphi} = -ie^{-i\varphi}\dot{\varphi},$$

whose correlation function

$$\langle \eta \eta_\tau^* \rangle = \langle \Omega \Omega_\tau e^{i\Delta\varphi} \rangle \quad (8.85)$$

can be replaced by $\langle \Omega \Omega_\tau \rangle$ for all τ, i.e.,

$$\langle \eta \eta_\tau^* \rangle = \langle \Omega \Omega_\tau (1 + i\Delta\varphi + \ldots) \rangle \to \langle \Omega \Omega_\tau \rangle. \quad (8.86)$$

This can be done because of (8.82) if $\tau \ll 1/\sigma_\Omega$, while if $\tau \gtrsim 1/\sigma_\Omega \gg \tau_{cor}$, both sides of (8.85) are near zero and hence differ only slightly. According to (8.86), the spectral density $S[\eta; \omega]$ coincides with the spectral density $s[\Omega; \omega]$ of the frequency $\Omega(t)$. But then

$$S[\eta; \omega] = \omega^2 S[z; \omega],$$

where $z = e^{-i\varphi}$ is the process with derivative η. Replacing $S[\eta; \omega]$ by $S[\Omega; \omega]$, we conclude that the process z, with correlation function $\langle e^{i\Delta\varphi} \rangle$, has spectral density

$$S[z; \omega] = \frac{1}{\omega^2} S[\Omega; \omega]. \quad (8.87)$$

To find the spectral density of

$$\langle xx_\tau \rangle = \tfrac{1}{2}A_0^2 \operatorname{Re}\{e^{i\omega_0 \tau} \langle z z_\tau^* \rangle\}$$

[cf. (8.33)], we need only multiply (8.87) by $\frac{1}{4}A_0^2$ and then recenter the spectral density at frequency ω_0:

$$S[x; \omega] = \tfrac{1}{4}A_0^2 S[z; |\omega - \omega_0|] = \tfrac{1}{4}A_0^2 \frac{S[\Omega; |\omega - \omega_0|]}{(\omega - \omega_0)^2} \left(|\omega - \omega_0| \sim \frac{1}{\tau_{cor}}\right). \quad (8.88)$$

Near the fundamental frequency ω_0, this expression becomes infinite, and then the previous formula (8.80) should be used instead.

It will be recalled from (8.72) that the spectral density $S[x;\omega]$ of the generated signal is proportional to the probability density of the frequency, in the case of large independent phase increments. Now, however, $S[x;\omega]$ involves the spectral density of the frequency fluctuations.

It is a characteristic feature of the case of small independent phase increments that the behavior of the spectral density for small deviations $\omega-\omega_0$ from the fundamental frequency is different from its behavior for large deviations. This twofold behavior does not occur for large increments (see Sec. 4).

Example 1. Consider the Gaussian frequency fluctuations considered earlier, leading to the spectral density (8.40). With the condition (8.76) and for small deviations $(\omega-\omega_0)^2 \ll \tau_{cor}^{-2}$, the argument the function $K_1(z)$ is much less than unity and hence can be replaced by its asymptotic expression $1/z$. This gives

$$S[x;\omega] = A_0^2 \frac{\sigma_\Omega^2 \tau_{cor}}{(\omega-\omega_0)^2 + \sigma_\Omega^4 \tau_{cor}^2}, \tag{8.89}$$

which agrees with (8.80), since

$$K_\Omega = 2\sigma_\Omega^2 \tau_{cor}$$

[as can be seen by integrating (8.30)].

Next suppose the condition (8.77) is combined with large deviations $(\omega-\omega_0)^2\tau_{cor}^2 \gg 1$. Then, neglecting $\sigma_\Omega^4\tau_{cor}^4$ compared to $(\omega-\omega_0)^2\tau_{cor}^2$, we find from (8.40) that

$$S[x;\omega] = A_0^2 \frac{\sigma_\Omega^2 \tau_{cor}^2}{|\omega-\omega_0|} K_1(\tau_{cor}|\omega-\omega_0|), \tag{8.90}$$

which agrees with (8.88), since the correlation coefficient (8.36) corresponds to the spectral density[1]

$$S[\Omega;\omega] = 4\sigma_\Omega^2 \tau_{cor}^2 \omega K_1(\tau_{cor}\omega).$$

[1] See I. M. Ryshik and I. S. Gradstein, *op. cit.*, formula (5.116), p. 250.

Example 2. For the Gaussian process with correlation function (8.41), the condition (8.76) for small independent increments takes the form $\Lambda \ll 1$, and guarantees the rapid convergence of the series (8.47). Retaining the first two terms of this series and neglecting quantities of order Λ^2, we obtain

$$S[x;\omega] = \frac{\Lambda\beta^3 A_0^2}{[(\omega-\omega_0)^2 + \Lambda^2\beta^2][(\omega-\omega_0)^2 + \beta^2]}. \qquad (8.91)$$

In particular, we have

$$S[x;\omega] = \frac{\Lambda\beta A_0^2}{(\omega-\omega_0)^2 + \Lambda^2\beta^2} \quad \text{for} \quad |\omega-\omega_0| \sim \Lambda\beta$$

and

$$S[x;\omega] = \frac{\Lambda\beta^3 A_0^2}{(\omega-\omega_0)^2[(\omega-\omega_0)^2 + \beta^2]} \quad \text{for} \quad |\omega-\omega_0| \sim \beta,$$

in complete agreement with (8.80) and (8.88), since

$$K_\Omega = 2\Lambda\beta, \quad S[\Omega;\omega] = \frac{4\Lambda\beta^3}{\omega^2+\beta^2}.$$

Example 3. Suppose the random process ζ, defined by formula (8.19) and figuring in (8.23), is essentially non-Gaussian ($\gamma\sigma \gtrsim \beta$). Then its correlation function coincides with the correlation function (7.37) of the process (7.35), and hence

$$\begin{aligned}\langle \Omega\Omega_\tau\rangle &= (\omega_0\mu_2)^2\left(\frac{\omega_0 M'\sigma}{2}\right)^2[\beta^2 R(\tau) + 2\gamma^2\sigma^2 R^2(\tau)] \\ &= (4\mu\mu_2\omega_0\sigma^2 A_0^{-2})^2\left[2R^2(\tau) + \left(\frac{\beta}{\gamma\sigma}\right)^2 R(\tau)\right]. \qquad (8.92)\end{aligned}$$

The form of the corresponding spectral density depends in an essential way on the concrete form of the correlation function $R(\tau)$ in (8.92). For example, if $R(\tau) = e^{-|\tau|/\tau_{cor}}$, then $R(\tau)$ and $R^2(\tau)$ have spectral densities

$$\frac{4\tau_{cor}}{\omega^2\tau_{cor}^2+1} \quad \text{and} \quad \frac{8\tau_{cor}}{\omega^2\tau_{cor}^2+4},$$

respectively. Therefore

$$S[\Omega;\omega] = 2(\mu\mu_2\omega_0)^2\left(\frac{2\sigma}{A_0}\right)^4\left[\frac{4\tau_{cor}}{\omega^2\tau_{cor}^2+4}+\left(\frac{\beta}{\gamma\sigma}\right)^2\frac{\tau_{cor}}{\omega^2\tau_{cor}^2+1}\right],$$

and hence, according to (8.88),

$$S[x;\omega] = \tfrac{1}{2}(\mu\mu_2 A_0^2)\left(\frac{2\sigma}{A_0}\right)^4\frac{\omega_0^2}{(\omega-\omega_0)^2}\left[\frac{4\tau_{cor}}{(\omega-\omega_0)^2\tau_{cor}^2+4}+\left(\frac{\beta}{\gamma\sigma}\right)^2\frac{\tau_{cor}}{(\omega-\omega_0)^2\tau_{cor}^2+1}\right]. \tag{8.93}$$

At the same time the discrete spectral line is described by the expression (8.80), where

$$K = (\mu\mu_2\omega_0)^2\left(\frac{2\sigma}{A_0}\right)^4\left[1+\left(\frac{\beta}{\gamma\sigma}\right)^2\right]\tau_{cor}. \tag{8.94}$$

As can be seen from the above discussion, almost all the power of the oscillations, equal to $\frac{1}{2}A_0^2$, is concentrated in the discrete spectral line with spectral density (8.80). The more broad-band fluctuations, whose frequencies fall in the range

$$|\omega-\omega_0| \sim \frac{1}{\tau_{cor}},$$

have small power. The order of magnitude of the spectral density in this range is $\frac{1}{2}A_0^2\sigma_\Omega^2\tau_{cor}^3$, and its total power can be estimated as $\frac{1}{2}A_0^2\sigma_\Omega^2\tau_{cor}^2$, a quantity which may turn out to be smaller than the contribution at these frequencies from the amplitude fluctuations (whose effect has been neglected).

6. Modulation of Self-Excited Oscillations by Noise

The slow fluctuations considered so far, which upset the stable operation of an oscillator and cause amplitude and phase "departures," are characterized by their small size, and hence lead to only small deviations from the stable values of the amplitude A_0 and frequency ω_0. However, in some cases, the oscillator is deliberately subjected to the action of slowly fluctuating noise of much greater intensity, e.g.,

for the purpose of obtaining noise-modulated oscillations (as in noise generators). If the correlation time of the modulating noise is greater than the relaxation time of the amplitude, then this case resembles that just considered, where the amplitude and frequency of the self-excited oscillations are modulated by ambient noise. The only difference is that the noise intensity is now large enough to produce deep modulation, i.e., a large fluctuational scatter

$$\mathbf{D}A \sim A_0^2. \tag{8.95}$$

The resulting oscillations are still described by equations (8.3) and (8.4), which take the form (8.20) and (8.21) if the modulating voltage fluctuations are applied to the grid of the oscillator, except that now we must set $u = u_g$ rather than $u = Du_a$ in the expression (8.19) for ζ.

To calculate the spectral density of the generated signal, we can no longer use formulas (8.32) and (8.33), and instead, we must now take account of the amplitude fluctuations:

$$x = A(\zeta)\cos(\omega_0 t + \varphi),$$

$$\langle x x_\tau \rangle = \tfrac{1}{2}\,\mathrm{Re}\left[\left\langle A(\zeta)A(\zeta_\tau)\exp\left\{i\int_{t_0}^{t_0+\tau}\Omega\,dt\right\}\right\rangle e^{i\omega_0\tau}\right]. \tag{8.96}$$

As before, there are two essentially different fluctuational processes, depending on the size of $\sigma_\Omega \tau_{cor}$ compared to unity. We begin with the case of large independent phase increments

$$\sigma_\Omega \tau_{cor} \gg 1. \tag{8.97}$$

The presence of the phase factor $e^{i\Delta\varphi}(\Delta\varphi \sim \sigma_\Omega \tau)$ causes the correlation function (8.96) to fall off sharply for $\tau > 1/\sigma_\Omega$, and in fact, (8.96) practically vanishes for $\tau \gg 1/\sigma_\Omega$. But the range $\tau \lesssim 1/\sigma_\Omega$ where (8.96) is appreciably nonzero is much narrower than the correlation time, because of (8.97), and hence the amplitude and frequency change very little in this range. Therefore the correlation function (8.96) takes the form

$$\langle x x_\tau \rangle = \tfrac{1}{2}\,\mathrm{Re}\left\{\langle A^2(\zeta)e^{i\Omega(\zeta)\tau}\rangle e^{i\omega_0\tau}\right\}, \tag{8.98}$$

where knowledge of the univariate distribution $w(\zeta)$ is sufficient to

calculate the average. Since

$$\int \langle A^2(\zeta) e^{-i\Omega(\zeta)\tau} \rangle e^{i\omega\tau}\, d\tau = 2\pi \langle A^2(\zeta)\, \delta[\omega - \Omega(\zeta)] \rangle \ .$$

it follows from (8.98) that

$$S[x;\omega] = \pi \langle A^2(\zeta)\, \delta[\omega - \omega_0 - \Omega(\zeta)] \rangle \,. \tag{8.99}$$

To evaluate the average in (8.99) [an integral involving the probability density $w(\zeta)$], we use the properties of the delta function. Suppose the equation

$$\Omega(\zeta) - z = 0$$

has r roots, which we denote by

$$\zeta = Q_1(z), \ldots, Q_r(z) \,.$$

Then the average can be written as the following sum over the different roots:

$$S[x;\omega] = \pi \sum_{j=1}^{r} A^2(\zeta)\, w(\zeta) \left| \frac{d\zeta}{d\Omega} \right|_{\zeta = Q_j(\omega - \omega_0)} \,. \tag{8.100}$$

If the function $\Omega(\zeta)$ has a single-valued inverse $\zeta = Q(\Omega)$, the sum (8.100) reduces to a single term, involving the probability density of the frequency:

$$S[x;\omega] = \pi A^2[Q(\omega - \omega_0)]\, w_\Omega(\omega - \omega_0) \,. \tag{8.101}$$

We now apply (8.100) to the specific equations (8.20) and (8.21). Dropping the term giving the nonrandom frequency shift, and using the symbols μ_2 and A_0 with their previous meanings, we can write these equations in the form

$$A = A_0 \sqrt{1 + \frac{\zeta}{\mu}} \,, \tag{8.102}$$

$$\dot{\varphi} = \omega_0 \mu_2 \zeta + \tfrac{29}{12} \omega_0 \zeta^2 \,. \tag{8.103}$$

To keep things simple, we replace the number 29 by 30 in (8.103). The function

$$\Omega(\zeta) = \omega_0 \mu_2 \zeta + \tfrac{5}{2} \omega_0 \zeta^2 \tag{8.104}$$

has a double-valued inverse

$$\zeta = Q_{1,2}(\Omega),$$
$$Q_{1,2}(\Omega) = \frac{1}{5}\left(-\mu_2 \pm \sqrt{\mu_2^2 + 10\frac{\Omega}{\omega_0}}\right). \tag{8.105}$$

Substituting (8.105) into the expression $\omega_0(\mu_2 + 5\zeta)$ for the derivative $d\Omega/d\zeta$ of (8.104), we find that both roots lead to the same value of

$$\left|\frac{d\zeta}{d\Omega}\right|_{\zeta = Q_{1,2}} = \frac{1}{\omega_0\sqrt{\mu_2^2 + 10\dfrac{\Omega}{\omega_0}}}.$$

Therefore (8.100) finally gives

$$S[x;\omega] = \pi\frac{A_0^2}{\mu\omega_0}\frac{1}{\sqrt{v}}\left[\frac{5\mu - \mu_2 + \sqrt{v}}{5} w_\zeta\left(\frac{-\mu_2 + \sqrt{v}}{5}\right)\right.$$
$$\left. + \frac{5\mu - \mu_2 - \sqrt{v}}{5} w_\zeta\left(\frac{-\mu_2 - \sqrt{v}}{5}\right)\right], \tag{8.106}$$

where

$$v = \mu_2^2 + 10\frac{\omega - \omega_0}{\omega_0} > 0.$$

This expression is valid for frequencies leading to positive values of v. The fluctuations have vanishing spectral density in the region where $v < 0$.

For strong external noise, producing deep modulation, we have $\langle\zeta^2\rangle \sim \mu^2$. Then the spectral density (8.106) of the generated signal lies in the frequency band $\omega_0 - \Delta\omega < \omega < \omega_0 + \Delta\omega$, of width

$$2\Delta\omega \sim \mu^2\omega_0. \tag{8.107}$$

We now turn to the case of small independent phase increments:

$$\sigma_\Omega\tau_{cor} \ll 1.$$

Then the spectral density of the generated signal has a twofold structure, i.e., there is a strong narrow spectral line together with weak fluctuations which are more broad-band. The discrete spectral line can

be investigated by the same method as in Sec. 5, since the amplitude fluctuations have no discrete component but lie in the band

$$|\omega - \omega_0| \sim \frac{1}{\tau_{cor}}. \tag{8.108}$$

The phase fluctuations considered in Sec. 5 also lie in this band, where, as noted on p. 215, their total power is of order $\frac{1}{2}A_0^2\sigma_\Omega^2\tau_{cor}^2$. At the same time, for deep modulation (8.95), the amplitude fluctuations, which lead to a correlation function of the form

$$\langle xx_\tau\rangle = \tfrac{1}{2}\langle AA_\tau\rangle \cos\omega_0\tau, \tag{8.109}$$

if it is assumed that the phase φ is constant, contribute power of order $\frac{1}{2}A_0^2$. In the present case, this power greatly exceeds that due to the phase fluctuations. Therefore, for deep modulation, we can neglect the effect of frequency fluctuations if we are interested in the range $|\omega-\omega_0|\sim 1/\tau_{cor}$ (the frequency fluctuations become important if $|\omega-\omega_0| \ll 1/\tau_{cor}$). This allows us to use formula (8.109), which leads at once to the following relation between $S[x;\omega]$ and the spectral density of the amplitude:

$$S[x;\omega] = \tfrac{1}{4}S[A;\omega-\omega_0]. \tag{8.110}$$

Since the amplitude is a zero-memory nonlinear transformation of the random function $\zeta(t)$ [see (8.102)], its spectral density can be calculated by the methods given in Chap. I.8. For example, if

$$\beta = 0, \quad \zeta = -\frac{4\mu}{A_0^2}u^2, \quad A = A_0\sqrt{1-4\frac{u^2}{A_0^2}},$$

we can use the next to the last formula on p. 1.236, obtaining the spectral density

$$S[A-\langle A\rangle;\omega] = 4A_0^2\sum_{m=1}^{\infty}\frac{1}{(2m)!}g_{1.2m}^2\left(\frac{A_0}{2\sigma}\right)\int_0^\infty R^{2m}(\tau)\cos\omega\tau\,d\tau. \tag{8.111}$$

The intermediate case of medium-sized independent phase increments $\sigma_\Omega\tau_{cor}\sim 1$ is the most difficult. Here formidable calculations lead to some results of no great generality or precision. There is an

exception, of course, namely the case of Gaussian frequency fluctuations $\Omega(t)$. Then we can use the general rule to immediately write the joint characteristic function of the random variables Ω, Ω_τ and $\Delta\varphi$:

$$\begin{aligned}\langle \exp\{i(u\Omega + v\Omega_\tau + w\Delta\varphi)\}\rangle &= \exp\{-\tfrac{1}{2}[\langle\Omega^2\rangle(u^2+v^2)\\ &\quad + 2\langle\Omega\Omega_\tau\rangle uv + \langle(\Delta\varphi)^2\rangle w^2 + 2\langle\Omega\Delta\varphi\rangle uw + 2\langle\Omega_\tau\Delta\varphi\rangle vw]\}\\ &= \langle \exp i\{(u\Omega + v\Omega_\tau)\}\rangle \exp\{-\tfrac{1}{2}\langle(\Delta\varphi)^2\rangle w^2\\ &\quad - \langle\Omega\Delta\varphi\rangle uw - \langle\Omega_\tau\Delta\varphi\rangle vw\}, \end{aligned} \tag{8.112}$$

where $\langle \exp\{i(u\Omega + v\Omega_\tau)\}$ is the two-dimensional characteristic function of Ω and Ω_τ. Let $A(\Omega)$ be the function giving the dependence of the amplitude on the frequency. Then the quantity $\langle A(\Omega)A(\Omega_\tau)e^{i\Delta\varphi}\rangle$ figuring in (8.96) can be expressed in terms of the characteristic function (8.112), i.e.,

$$\begin{aligned}\langle A(\Omega)A(\Omega_\tau)e^{i\Delta\varphi}\rangle = \frac{1}{(2\pi)^2}\int\int\int\int A(\Omega')A(\Omega'_\tau)e^{-iu\Omega' - iv\Omega'_\tau}\\ \times\langle e^{iu\Omega + iv\Omega_\tau + i\Delta\varphi}\rangle\, d\Omega'\, d\Omega'_\tau\, du\, dv, \end{aligned} \tag{8.113}$$

where the validity of (8.113) can easily be proved by noting that

$$\begin{aligned}\frac{1}{2\pi}\int e^{-iu(\Omega' - \Omega)}\, du &= \delta(\Omega' - \Omega),\\ \frac{1}{2\pi}\int e^{-iv(\Omega'_\tau - \Omega_\tau)}\, dv &= \delta(\Omega'_\tau - \Omega_\tau). \end{aligned} \tag{8.114}$$

Substituting (8.112) into (8.113), integrating with respect to u, v, Ω', Ω'_τ and using (8.114), we obtain

$$\langle A(\Omega)A(\Omega_\tau)e^{i\Delta\varphi}\rangle = e^{-\langle(\Delta\varphi)^2\rangle/2}\langle A(\Omega + i\langle\Omega\Delta\varphi\rangle)A(\Omega_\tau + i\langle\Omega_\tau\Delta\varphi\rangle)\rangle. \tag{8.115}$$

Thus the problem reduces to finding the correlation function of a nonlinear zero-memory transformation $A = A(\Omega + m)$ of the random process $\Omega(t)$, where $m = i\langle\Omega\Delta\varphi\rangle$. Since $A(\Omega + m)$ is an analytic function of m, we can carry out the calculation by Rice's method, say (see Chap. I.8, Sec. 4), regarding m as a real constant. Then, after calculating the correlation function

$$\langle xx_\tau\rangle = \tfrac{1}{2}e^{-\langle(\Delta\varphi)^2\rangle/2}\,\mathrm{Re}\,\{\langle A(\Omega + m)A(\Omega_\tau + m)\rangle e^{i\omega_0\tau}\}, \tag{8.116}$$

m and $\langle(\Delta\varphi)^2\rangle$ must be replaced by their actual values

$$m = i\langle\Omega\Delta\varphi\rangle = i\langle\Omega_\tau\Delta\varphi\rangle = i\sigma_\Omega^2\int_0^\tau R(\tau')\,d\tau',$$
$$\langle(\Delta\varphi)^2\rangle = 2\sigma_\Omega^2\int_0^\tau(\tau-\tau')R(\tau')\,d\tau'. \tag{8.117}$$

CHAPTER 9

Synchronization of an Oscillator in the Presence of Noise[1]

There are many problems of radio engineering (requiring, for example, large channel capacity or high noise immunity) which can only be solved by using oscillators with a high degree of frequency stability. However, oscillators are always subject to the action of noise (e.g., fluctuations in anode current due to shot noise, fluctuations in external voltages, etc.) which decrease the frequency stability and increase the bandwidth of the generated signal. An effective means of achieving self-excited oscillations with high frequency stability is the use of *external synchronization*, i.e., the given oscillator is subjected to a synchronizing voltage produced by another oscillator which has greater frequency stability despite its smaller power. The effect of the synchronizing voltage is to "lock in" and stabilize the frequency of the primary oscillation.

Despite the use of a synchronizing voltage, the presence of noise still manages to produce departures from a completely synchronous operating regime. In the first place there are incessant small fluctuational deviations of the phase from its synchronous value. Secondly, there are occasionally large phase deviations, comparable to $\pi/2$. Thirdly, the phase sometimes (albeit infrequently) jumps by an integral number of periods. These uncompensated phase jumps accumulate and cause

[1] The author would like to thank R. V. Khokhlov for advice which was helpful in obtaining a number of results presented in this chapter.

the mean frequency of the oscillations to differ from the synchronizing frequency. They also lead to the occurrence of irreversible phase diffusion, thereby producing frequency instability of a diffusional character.

The first effect (small phase fluctuations) can be treated by using the linearization method. The other effects are essentially nonlinear, and are most conveniently investigated by using the apparatus of Markov processes, which, in dealing with nonlinear problems, has the great computational advantage of allowing one to determine the stationary distribution by quadratures.

1. The Basic Equations. Small Deviations from the Synchronous Regime in the Linear Approximation

If the synchronization voltage is applied to the inductive branch, then, according to (4.17), the operation of the oscillator is described by the equation

$$\ddot{x} + \omega_0^2 x = \omega_0^2\left[\omega_s MF\left(\frac{\dot{x}}{\omega_s}\right) - RC\dot{x}\right] - \omega_0^2\omega_s^2 MCU_0 \sin \omega_s t + \omega_0^2\omega_s M I_f . \tag{9.1}$$

in terms of the voltage $x=\omega_s M(I-I_=)$. In the case where the noise is not due to shot noise but to a fluctuating voltage $u_b(t)$ applied to the inductive branch, $\omega_0^2\omega_s M I_f$ should be replaced by $\omega_0^2\omega_s MCu_b$.

The next step is to write (9.1) in standard form. Unlike (4.30), we now define the amplitude and phase in terms of the synchronization frequency ω_s instead of the resonant frequency ω_0 of the tank circuit. This gives

$$x = A\cos(\omega_s t + \varphi), \quad \dot{x} = -\omega_s A \sin(\omega_s t + \varphi) \tag{9.2}$$

instead of (4.30), and hence

$$\dot{A} = -\frac{\sin(\omega_s t + \varphi)}{\omega_s}(\ddot{x} + \omega_s^2 x), \quad \dot{\varphi} = -\frac{\cos(\omega_s t + \varphi)}{\omega_s A}(\ddot{x} + \omega_s^2 x) \tag{9.3}$$

[cf. (4.32)]. Using our previous approximation (4.23) for the tube

characteristic and introducing the notation

$$\varepsilon = \frac{\omega_0^2}{\omega_s}(MS - RC), \quad A_0^2 = 4\frac{MS - RC}{M\gamma},$$
$$\Delta = \frac{\omega_0^2 - \omega_s^2}{2\omega_s} \approx \omega_0 - \omega_s, \quad E = \omega_0^2 MCU_0 = \frac{M}{L}U_0, \tag{9.4}$$

we can write (9.1) in the form

$$\ddot{x} + \omega_s^2 x = -2\Delta\omega_s x + \varepsilon\omega_s\dot{x}\left[1 - \frac{4}{3A_0^2}\left(\frac{\dot{x}}{\omega_s}\right)^2\right] - \omega_s^2 E\sin\omega_s t - \omega_s^2\eta(t), \tag{9.5}$$

suitable for substitution into (9.3). Here $\eta(t)$ is the random process

$$\eta = -\frac{\omega_0^2}{\omega_s}MI_f \quad \text{or} \quad \eta = -\frac{\omega_0^2}{\omega_s}MC\dot{u}_b, \tag{9.6}$$

depending on the nature of the noise. After making the indicated substitution and dropping oscillatory terms in the resulting standard equations, we arrive at the following truncated equations in the first approximation:

$$\dot{A} = \frac{\varepsilon\omega_s}{2}\left(1 - \frac{A^2}{A_0^2}\right)A + \frac{\omega_s E}{2}\cos\varphi + \omega_s\eta\sin(\omega_s t + \varphi),$$
$$\dot{\varphi} = \Delta - \frac{\omega_s E}{2A}\sin\varphi + \frac{\omega_s}{A}\eta\cos(\omega_s t + \varphi). \tag{9.7}$$

The fluctuational terms $\eta\sin\Phi$ and $A^{-1}\eta\cos\Phi$ appearing in (9.7) can be simplified by the same method as in Chap. 4, Sec. 4, if the correlation time of the process $\eta(t)$ is sufficiently small compared to the relaxation time:

$$\tau_{cor} \ll \frac{1}{\varepsilon\omega_0}. \tag{9.8}$$

(If this condition is satisfied, we can apply stochastic methods and the Fokker-Planck equation to the present problem.) As a result, we

obtain the simplified equations

$$\dot{A} = \frac{\varepsilon\omega_s}{2}\left(1 - \frac{A^2}{A_0^2}\right)A + \frac{\omega_s E}{2}\cos\varphi + \frac{K}{2A} + \xi_1,$$
$$\dot{\varphi} = \Delta - \frac{\omega_s E}{2A}\sin\varphi + \frac{\xi_2}{A}, \tag{9.9}$$

where ξ_1 and ξ_2 can be regarded as independent zero-mean delta-correlated Gaussian random functions:

$$\langle\xi_{1,2}\rangle = 0, \quad \langle\xi_1\xi_{2\tau}\rangle = 0,$$
$$\langle\xi_1\xi_{1\tau}\rangle = \langle\xi_2\xi_{2\tau}\rangle = K\,\delta(\tau), \quad K = \frac{\omega_s^2}{2}\kappa_\eta(\omega_s). \tag{9.10}$$

If ω_s coincides with ω_0, then the random functions ξ_1 and ξ_2 are connected with the random functions $\zeta, \eta, \zeta', \eta'$ appearing in equations (4.66), (4.67) and (7.4) by the relations

$$\xi_1 = \varepsilon\omega_0\eta = A_0\eta', \quad \xi_2 = \varepsilon\omega_0\zeta = A_0\zeta'.$$

In deriving (9.9), we assumed that the synchronizing excitation is of sinusoidal form. However, exactly the same equations describe the case where the external excitation is a periodic function of arbitrary form, e.g., a sequence of pulses, provided that the frequency of the excitation is near the fundamental frequency of the oscillations (synchronization at multiple or fractional frequencies will not be considered). Any periodic synchronizing signal $f_s(t)$ can be replaced by its Fourier expansion

$$f_s(t) = E_0 + E_1\cos\omega_s t + E_1'\sin\omega_s t + E_2\cos 2\omega_s t + E_2'\sin 2\omega_s t + \ldots$$

Substituting this expansion into (9.3) and dropping oscillatory terms to a first approximation, we find that all terms except $E_1\cos\omega_s t$ and $E_1'\sin\omega_s t$ disappear, leaving

$$E_1\cos\omega_s t + E_1'\sin\omega_s t = \sqrt{E_1^2 + E_1'^2}\sin(\omega_s t + \psi).$$

Moreover, the phase ψ depends on the choice of the time origin, and can be set equal to zero. Therefore, insofar as its synchronizing action is concerned, the function $f_s(t)$ is completely equivalent (at least, to a

first approximation) to the sinusoid $E\sin\omega_s t$ considered above, provided we now write

$$E=\left\{\left[\frac{1}{\pi}\int_0^{2\pi} f_s(t)\cos\omega_s t\,d(\omega_s t)\right]^2+\left[\frac{1}{\pi}\int_0^{2\pi} f_s(t)\sin\omega_s t\,d(\omega_s t)\right]^2\right\}^{1/2}.$$

In this sense, the results obtained below are valid for a synchronizing signal of arbitrary form. Thus, if the synchronization is produced by pulses of height E_0, occupying the fraction Θ of a period, the appropriate choice is

$$E=\frac{2}{\pi}E_0\sin\pi\Theta.$$

The basic system of equations (9.9) is hard to solve in its most general form. However, it can be used to obtain a number of important special results. First we consider the case of no random excitation at all ($\xi_1=\xi_2=0$, $K=0$). Then there will be a synchronous regime with constant values A_1, φ_1 of the amplitude and phase, provided a certain condition for "locking in" is satisfied [see (9.14)]. Equations for A_1 and φ_1 can be obtained from (9.9) by setting $\dot{A}$ and $\dot{\varphi}$ equal to zero. This gives

$$\begin{aligned}\varepsilon(a_1^2-1)a_1&=\frac{E}{A_0}\cos\varphi_1,\\ \frac{2\Delta}{\omega_s}a_1&=\frac{E}{A_0}\sin\varphi_1,\end{aligned}\tag{9.11}$$

where $a_1=A_1/A_0$. Combining the two equations, we obtain

$$\left[(a_1^2-1)^2+\left(\frac{2\Delta}{\varepsilon\omega_s}\right)^2\right]a_1^2=\left(\frac{E}{\varepsilon A_0}\right)^2.\tag{9.12}$$

Solving this cubic equation in a_1^2, we can find stationary values A_1, φ_1 of the amplitude and phase for every case of interest. Equation (9.12) is easily studied graphically by writing it in the form

$$x(x-1)^2=\left(\frac{E}{\varepsilon A_0}\right)^2-\left(\frac{2\Delta}{\varepsilon\omega_s}\right)^2 x\quad(x=a_1^2).\tag{9.13}$$

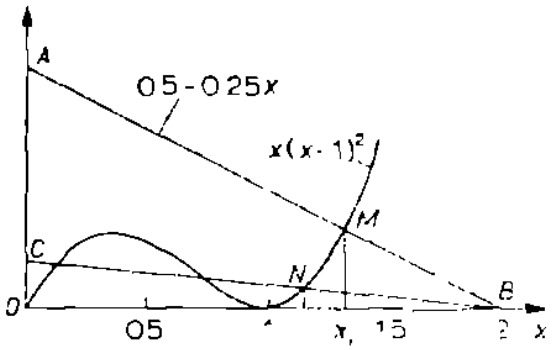

Fig. 22. Graphical determination of the stationary amplitude a_1 [illegible]. The line AB corresponds to the parameters $(E/\varepsilon A_0)^2 = \frac{1}{2}$, $(2\Delta/\varepsilon\omega_s)^2 = \frac{1}{4}$, and CD to the parameters $(E/\varepsilon A_0)^2 = \frac{1}{10}$, $(2\Delta/\varepsilon\omega_s)^2 = \frac{1}{20}$.

In Figure 22, the graph of the function $x(x-1)^2$ is intersected by the line corresponding to the right-hand side of (9.13) and determined by the parameters $E/\varepsilon A_0$ and $2\Delta/\varepsilon\omega_s$. If there are three points of intersection instead of one (as in the case of the line CB), it can be shown by considerations resembling those on pp. 136–138 that the stable amplitude corresponds to the point N with the largest abscissa.

A synchronous regime is possible only if the "mismatch" Δ does not exceed a certain value depending on the amplitude E. It follows from the second of the equations (9.11) that the condition for "locking in" takes the form

$$\Delta < \frac{\omega_s E}{2A_1} \quad \text{or} \quad \Delta < \frac{\Delta_s}{a_1}, \tag{9.14}$$

where $\Delta_s = \omega_s E/2A_0$ is a system parameter, called the *synchronization bandwidth*. Assuming that this condition is satisfied, we now study the case of small fluctuational deviations from the synchronous regime.

The expressions on the right in (9.9) without the fluctuational terms ξ_1, ξ_2 take small values near the point $A = A_1$, $\varphi = \varphi_1$, since they vanish at this point. Therefore we can expand these expressions in Taylor series near $A = A_1$, $\varphi = \varphi_1$, retaining only linear terms. In calculating the corresponding first partial derivatives with respect to A and φ, we use (9.11). Since linearization is possible only in the case of weak noise, the fluctuational term ξ_2/A can be replaced by ξ_2/A_1. Moreover, the term $K/2A$ can be dropped, since it is of the same order as the omitted

quadratic terms. In this way, we arrive at the following linearized equations

$$\begin{aligned} \delta\dot{A} &= -\varepsilon\omega_s \frac{3a_1^2-1}{2}\delta A - \Delta A_1\,\delta\varphi + \xi_1, \\ \delta\dot{\varphi} &= \frac{\Delta}{A_1}\delta A - \varepsilon\omega_s \frac{a_1^2-1}{2}\delta\varphi + \frac{\xi_2}{A_1}, \end{aligned} \tag{9.15}$$

where

$$\delta A = A - A_1, \quad \delta\varphi = \varphi - \varphi_1.$$

After introducing the abbreviated notation

$$\delta A = y, \quad A_1\delta\varphi = z, \quad \varepsilon\omega_s(3a_1^2-1) = 2\delta, \quad \varepsilon\omega_s(a_1^2-1) = 2\delta', \tag{9.16}$$

the equations (9.15) become

$$\begin{aligned} \dot{y} &= -\delta y - \Delta z + \xi_1, \\ \dot{z} &= \Delta y - \delta' z + \xi_2. \end{aligned} \tag{9.17}$$

The system (9.17) can be solved by ordinary methods, and the initial conditions have no effect if we are interested only in the stationary regime. This corresponds to regarding the perturbations ξ_1 and ξ_2 as turned on in the very remote past ($t = -\infty$). Applying the spectral method, we replace the differentiation operator in (9.17) by $i\omega$, and solve the resulting equations

$$(i\omega + \delta)y + \Delta z = \xi_1, \quad -\Delta y + (i\omega + \delta')z = \xi_2$$

as an algebraic system for y and z. This gives

$$\begin{aligned} y &= \frac{(i\omega + \delta')\xi_1 - \Delta\xi_2}{Z(i\omega)} \equiv L_{y1}\xi_1 + L_{y2}\xi_2, \\ z &= \frac{\Delta\xi_1 + (i\omega + \delta)\xi_2}{Z(i\omega)} \equiv L_{z1}\xi_1 + L_{z2}\xi_2, \end{aligned}$$

where

$$Z(i\omega) = (i\omega + \delta)(i\omega + \delta') + \Delta^2$$

and the different L's are abbreviations for certain complex expressions. It follows from the usual rules governing spectral densities of linear

transformations of stationary processes that

$$\kappa_y = |L_{y1}|^2 \kappa_{\xi 1} + |L_{y2}|^2 \kappa_{\xi 2},$$
$$\kappa_z = |L_{z1}|^2 \kappa_{\xi 1} + |L_{z2}|^2 \kappa_{\xi 2},$$
$$\kappa_{yz} = \kappa_{zy}^* = L_{y1} L_{z1}^* \kappa_{\xi 1} + L_{y2} L_{z2}^* \kappa_{\xi 2} \quad (\kappa_{\xi 1} = \kappa_{\xi 2} = K)$$

where we use the fact that $\xi_1(t)$ and $\xi_2(t)$ are independent. In expanded form, we have

$$\kappa_y(\omega) = \tfrac{1}{2} S[y;\omega] = \frac{\omega^2 + \delta'^2 + \Delta^2}{\sqrt{(\delta\delta' + \Delta^2 - \omega^2)^2 + \omega^2(\delta + \delta')^2}} K,$$
$$\kappa_z(\omega) = \tfrac{1}{2} S[z;\omega] = \frac{\omega^2 + \delta^2 + \Delta^2}{\sqrt{(\delta\delta' + \Delta^2 - \omega^2)^2 + \omega^2(\delta + \delta')^2}} K, \tag{9.18}$$
$$\kappa_{yz}(\omega) = \tfrac{1}{2} S[y,z;\omega] = \frac{\delta' - \delta + 2i\omega}{\sqrt{(\delta\delta' + \Delta^2 - \omega^2)^2 + \omega^2(\delta + \delta')^2}} \Delta K.$$

Explicit expressions for the correlation functions corresponding to the spectral densities (9.18) can easily be found by taking inverse Fourier transforms. In fact, spectral densities like (9.18) have already been considered in Chap. I.2. Using entries 5 and 6 of Table 1, p. I.25, we see that the correlation functions of the processes $y(t)$ and $z(t)$ can be expressed in terms of the functions

$$e^{-\gamma|\tau|} \cos \nu\tau, \quad e^{-\gamma|\tau|} \sin \nu\tau, \tag{9.19}$$

where the quantities γ and ν satisfy the relations

$$\gamma^2 + \nu^2 = \delta\delta' + \Delta^2, \quad 2\gamma = \delta + \delta'$$

and hence equal

$$\gamma = \varepsilon\omega_s(a_1^2 - \tfrac{1}{2}),$$
$$\nu = \sqrt{\Delta^2 - \left(\frac{\delta - \delta'}{2}\right)^2} = \sqrt{\Delta^2 - \left(\frac{\varepsilon\omega_s}{2} a_1\right)^2}. \tag{9.20}$$

Forming the linear combinations of (9.19) which lead to (9.18), we easily obtain the correlation functions

$$k_y(\tau) = \frac{K}{4\gamma} \frac{1}{\delta\delta' + \Delta^2} e^{-\gamma|\tau|} \left\{ (\delta'^2 + \delta\,\delta' + 2\Delta^2) \cos \nu\tau \right.$$
$$\left. - 2(\delta - \delta') \frac{\delta'\gamma}{\nu} \sin \nu|\tau| \right\},$$

$$k_z(\tau) = \frac{K}{4\gamma}\frac{1}{\delta\delta' + \Delta^2} e^{-\gamma|\tau|}\left\{(\delta^2 + \delta\delta' + 2\Delta^2)\cos\nu\tau + 2(\delta - \delta')\frac{\delta\gamma}{\nu}\sin\nu|\tau|\right\},$$

$$k_{yz}(\tau) = \frac{K}{4\gamma}\frac{2\Delta}{\delta\delta' + \Delta^2}\left(\delta' - \delta - 2\frac{d}{d\tau}\right)\left\{e^{-\gamma|\tau|}\left(\cos\nu\tau + \frac{\gamma}{\nu}\sin\nu|\tau|\right)\right\}. \tag{9.21}$$

Setting $\tau = 0$ in (9.21), we obtain the variance and correlation coefficient of the amplitude and phase fluctuations:

$$\langle y^2\rangle = \frac{K}{2\gamma}\frac{\delta'\gamma + \Delta^2}{\delta\delta' + \Delta^2}, \quad \langle z^2\rangle = A_1^2\langle(\delta\varphi)^2\rangle = \frac{K}{2\gamma}\frac{\delta\gamma + \Delta^2}{\delta\delta' + \Delta^2},$$
$$\langle yz\rangle = \frac{\Delta K}{2\gamma}\frac{\delta' - \delta}{\delta\delta' + \Delta^2}, \quad R_{yz} = \Delta\frac{\delta' - \delta}{[(\delta'\gamma + \Delta^2)(\delta\gamma + \Delta^2)]^{1/2}}. \tag{9.22}$$

The results just obtained contain the quantities defined by (9.16) and (9.20), and can, of course, be written in other ways (we omit the details). However, it should be noted that the amplitude scatter is always less than the scatter in the perpendicular direction, i.e.,

$$\langle(\delta A)^2\rangle < A_1^2\langle(\delta\varphi)^2\rangle.$$

Correspondingly, the system always has more "stiffness" with respect to the amplitude (the coordinate y) than with respect to the coordinate z ($\delta > \delta'$). Writing the cross-correlation coefficient in the form

$$R_{yz} = -\varepsilon\omega_s a_1^2\Delta(\delta\delta'\gamma^2 + 2\Delta^2\gamma^2 + \Delta^4)^{-1/2},$$

we see that its sign is determined by that of the mismatch Δ.

The physical picture is particularly simple if $\Delta = 0$ (no mismatch). Then the amplitude and phase fluctuations are independent, since the system (9.17) splits into two uncoupled equations, which clearly describe exponentially correlated processes $y(t)$ and $z(t)$. As we know from formula (I.4.217), the variance of such a process equals the intensity of the random impacts divided by twice the "stiffness." In fact, setting $\Delta = 0$ in (9.22), we obtain

$$\langle y^2\rangle = \frac{K}{2\delta}, \quad \langle z^2\rangle = \frac{K}{2\delta'}, \quad \langle yz\rangle = 0. \tag{9.23}$$

In using this formula, it helps to bear in mind the approximate relations implied by (9.12) for the extreme cases of small and large amplitudes of the synchronizing signal. Thus

$$a_1^2 \approx 1 + \frac{E}{\varepsilon A_0}, \quad 2\delta \approx 2\varepsilon\omega_s, \quad 2\delta' \approx \omega_s \frac{E}{A_0} = 2\Delta_s \tag{9.24}$$

if $E/\varepsilon A_0 \ll 1$, while

$$a_1^2 \approx \left(\frac{E}{\varepsilon A_0}\right)^{2/3}, \quad 2\delta \approx 3\varepsilon\omega_s\left(\frac{E}{\varepsilon A_0}\right)^{2/3}, \quad 2\delta' \approx \varepsilon\omega_s\left(\frac{E}{\varepsilon A_0}\right)^{2/3} \tag{9.25}$$

if $E/\varepsilon A_0 \gg 1$.

Clearly, if (9.24) holds, the amplitude stiffness is the same as that of an autonomous oscillator. The phase stiffness is much smaller, leading to a large phase scatter, in fact, large enough to cast doubt on the applicability of the linearization method. Since the linearization method is valid only if $\langle(\delta\varphi)^2\rangle \ll 1$, i.e., if

$$\langle z^2\rangle \approx \frac{KA_0}{\omega_s E} \ll A_1^2,$$

it ceases to be applicable if

$$E \lesssim \frac{\omega_s \kappa_\eta(\omega_s)}{2A_0} \tag{9.26}$$

[cf. (9.10)]. As the synchronizing excitation is increased, the stiffness with respect to both amplitude and phase increases, with the phase stiffness increasing more rapidly, so that it almost catches up with the amplitude stiffness. In other words, the phase scatter decreases more rapidly than the amplitude scatter, and eventually approaches the amplitude scatter. In fact, according to (9.25), if $E/\varepsilon A_0 \gg 1$, the standard deviation of the phase is only $\sqrt{3}$ times larger than that of the relative amplitude $\delta A_1/A_1$.

Next we calculate the correlation function and spectral density of the generated signal $x = A\cos(\omega_s t + \varphi)$. If the linearization method is applicable, we are justified in writing

$$\begin{aligned} x &= (A_1 + y)\cos\left(\omega_s t + \varphi_1 + \frac{z}{A_1}\right) \\ &\approx A_1\cos(\omega_s t + \varphi_1) + y\cos(\omega_s t + \varphi_1) - z\sin(\omega_s t + \varphi_1), \end{aligned} \tag{9.27}$$

just as we linearized the basic equations (9.9) themselves. Multiplying (9.27) by the same expression taken at a different time and averaging the result, we obtain the correlation function of the signal, involving the correlation functions of $y(t)$ and $z(t)$ which have already been calculated. If the process x is handled *coherently*, under circumstances where the phase of the generated signal can be compared with the phase of the synchronizing signal, then x is a periodic nonstationary process (cf. Chap. I.5, Sec. 4), whose correlation function (obtained by the use of coherent operations) not only depends on the time difference τ but also varies periodically as both times are shifted by the same amount along the time axis. On the other hand, if x is handled *incoherently*, we can carry out an extra average over the indeterminate initial phase, which can be regarded as a uniformly distributed random variable. The resulting process, with a random initial phase, is now stationary, and its correlation function $\langle xx_\tau\rangle$ depends only on the time difference τ.

We can find $\langle xx_\tau\rangle$ most simply by applying the method described in Chap. I.7, based on the use of the complex random process

$$\begin{aligned} Z &= A_1 + y + iz, \\ x &= \operatorname{Re}\{Ze^{i(\omega_s t+\varphi_1)}\}. \end{aligned} \tag{9.28}$$

The correlation functions of the real and complex processes are connected by the relation

$$\langle xx_\tau\rangle = \tfrac{1}{2}\operatorname{Re}\{\langle ZZ_\tau^*\rangle e^{-i\omega_s\tau}\} \tag{9.29}$$

[cf. (I.7.16)], and their spectral densities by the relation

$$\begin{aligned} S[x;\omega] &= \tfrac{1}{4}S[Z;\omega-\omega_s] + \tfrac{1}{4}S^*[Z;\omega+\omega_s] \\ &\approx \tfrac{1}{4}S[Z;\omega-\omega_s] \quad \text{for} \quad \omega_s > 0, \end{aligned}$$

i.e.,

$$S[x;\omega] = \tfrac{1}{2}\kappa_Z(|\omega|-\omega_s). \tag{9.30}$$

Taking account of (9.28) and the expressions (9.18), (9.21) found above, we can determine the statistical characteristics of the complex process $Z(t)$:

$$\begin{aligned} \langle ZZ_\tau^*\rangle &= A_1^2 + \langle yy_\tau\rangle + \langle zz_\tau\rangle - i\langle yz_\tau - zy_\tau\rangle, \\ \kappa_Z(\omega) &= 2\pi A_1^2\,\delta(\omega) + \kappa_y(\omega) + \kappa_z(\omega) - i[\kappa_{yz}(\omega) - \kappa_{yz}^*(\omega)] \\ &= 2\pi A_1^2\,\delta(\omega) + \frac{2\omega^2 + \delta'^2 + \delta^2 + 2\Delta^2 + 4\omega\Delta}{(\delta\delta' + \Delta^2 - \omega^2)^2 + \omega^2(\delta+\delta')^2}K. \end{aligned} \tag{9.31}$$

The expressions (9.31) can then be substituted into (9.29) and (9.30). Thus for the spectral density we obtain

$$S[x;\omega] = \pi A_1^2\,\delta(|\omega| - \omega_s) + \frac{(|\omega| - \omega_s + \Delta)^2 + \frac{1}{2}(\delta'^2 + \delta^2)}{[\delta\delta' + \Delta^2 - (|\omega| - \omega_s)^2]^2 + (|\omega| - \omega_s)^2(\delta + \delta')^2}\,K. \tag{9.32}$$

Here, just as in Chap. 6, the presence of correlation between the amplitude and phase fluctuations causes the spectral density to be asymmetric with respect to the fundamental frequency ω_s (the numerator of the second term on the right is symmetric with respect to $\omega_s - \Delta$, while the denominator is symmetric with respect to ω_s).

Setting $\tau = 0$ in (9.29) and in the first of the equations (9.31), and taking account of (9.22), we obtain

$$\langle x^2\rangle = \tfrac{1}{2}(A_1^2 + \langle y^2\rangle + \langle z^2\rangle) = \tfrac{1}{2}A_1^2 + \frac{K}{2\gamma}\,\frac{\gamma^2 + \Delta^2}{\delta\delta' + \Delta^2}. \tag{9.33}$$

This shows that the power of the fluctuations with a continuous spectral density, described by the second term in the right-hand side of (9.32), makes up only

$$\frac{K}{\gamma A_1^2}\,\frac{\gamma^2 + \Delta^2}{\delta\delta' + \Delta^2} \times 100\% \tag{9.34}$$

of the discrete spectral line at the synchronization frequency, an amount which is always small if the linearization method is applicable.

A tentative estimate of the width of the continuous spectral band in (9.32) due to the fluctuations is given by $2\gamma = \varepsilon\omega_s(2a_1^2 - 1)$ or even $2\varepsilon\omega_s a_1^2$. Then the ratio of the bandwidth of the continuous part of the spectral density to the effective width of the synchronization band Δ_s/a_1 is

$$2\frac{\varepsilon\omega_s}{\Delta_s}a_1^3. \tag{9.35}$$

For the linearization method to be applicable, it is necessary that the fluctuational deviations $\langle(\delta A)^2\rangle/A_1^2$ and $\langle(\delta\varphi)^2\rangle$ be small compared to unity. However, these conditions are not sufficient, particularly in the case of the phase when the mismatch is large and approaches the boundary of the synchronization band. Then the synchronization

can be destroyed if the phase differs from that of the synchronous regime by amounts much smaller than π. If $\delta_0\varphi$ is such a deviation, a necessary condition for the validity of the results obtained above is that the inequality

$$\langle(\delta\varphi)^2\rangle \ll (\delta_0\varphi)^2 \tag{9.36}$$

be satisfied. Fulfillment of this condition is hampered further by the fact that the phase stiffness decreases as the boundary of the synchronization band is approached, thereby increasing the amount of phase scatter.

Thus we see that even when the noise level is low, the linearization method ceases to be applicable if the amplitude of the synchronizing excitation (9.26) becomes small or if the mismatch approaches the boundary of the synchronization band [thereby violating (9.36)]. To study these cases, we must use the nonlinear theory. Moreover, in the linearization approximation nothing at all can be said about the problem of phase jumps equal to an integral number of periods. In fact, it has been tacitly assumed that any fluctuational deviation, regardless of its size, is eliminated by the representative point returning to its original position. Actually, however, if the deviation is small compared to π, the representative point can approach the synchronous regime again "from the other side," by describing a closed loop around the origin of coordinates, as a result of which the phase changes by 2π. This is another effect that can only be understood in the context of the nonlinear theory, to which we now turn our attention.

2. The Stationary Phase Distribution and the Mean Frequency

Returning to the nonlinear equations (9.9), we assume that the inequality

$$\frac{E}{A_0} \ll \varepsilon, \quad \text{i.e.,} \quad \varepsilon\omega_s \gg \Delta_s \tag{9.37}$$

is satisfied. Then the first and second equations in the system (9.9) can be treated separately. In fact, if (9.37) holds, the amplitude changes much more rapidly than the phase, and hence the amplitude manages

to establish an equilibrium amplitude distribution $w(A|\varphi)$ for every value of the phase φ. To find $w(A|\varphi)$, we use the method of the Fokker-Planck equation (Chap. 5, Sec. 1), regarding the phase as constant in the first of the equations (9.9). Writing down the appropriate Fokker-Planck equation and using the stationary condition $\dot{w}(A|\varphi)=0$ for the conditional distribution, we obtain

$$w(A|\varphi)=\frac{A}{N_\varphi}\exp\frac{2}{K}\left\{\varepsilon\omega_s\frac{A^2}{4}\left(1-\frac{A^2}{2A_0^2}\right)+\frac{\omega_s E}{2}A\cos\varphi\right\}. \tag{9.38}$$

where N_φ is a normalization factor. The presence of the terms $\frac{1}{2}\omega_s EA\cos\varphi$ in (9.38) causes the most probable value of the amplitude to be shifted by an amount approximately equal to $(E/2\varepsilon)\cos\varphi$. Because of (9.37), this shift is much smaller than A_0, and neglecting it, we obtain the probability density

$$w(A|\varphi)=\frac{A}{N_\varphi}\exp\left\{\frac{\varepsilon\omega_s}{4K}A^2\left(2-\frac{A^2}{A_0^2}\right)\right\}=\frac{A}{N}\exp\left\{-\frac{\varepsilon\omega_s}{4K}A_0^2\left(\frac{A^2}{A_0^2}-1\right)^2\right\}, \tag{9.39}$$

which coincides with (5.8). Thus, in the case of a comparatively narrow synchronization band, when the condition (9.37) holds, the amplitude distribution is practically the same as the distribution of an autonomous oscillator (at any noise level).

If the noise intensity is small, i.e.,

$$K\ll\varepsilon\omega_s A_0^2\approx A_0^2\delta, \tag{9.40}$$

the distribution (9.39) is close to being Gaussian for small deviations $A-A_0$:

$$w(A|\varphi)\approx\frac{A_0}{N}\exp\left\{-\frac{\varepsilon\omega_s}{K}(A-A_0)^2\right\}. \tag{9.41}$$

This approximation agrees with the linear approximation, in which all distribution laws are assumed to be Gaussian. In fact, using (9.37) and (9.24), we find that (9.22) leads to the variance

$$\langle(\delta A)^2\rangle\approx\frac{K}{2\delta}\approx\frac{K}{2\varepsilon\omega_s}.$$

Therefore the amplitude distribution obtained by the linearization method is actually the same as (9.41) [cf. (5.39)].

We now turn to the second equations (9.9), describing the phase behavior. If (9.40) holds, the relative amplitude scatter is small, i.e.,

$$\langle (A - A_0)^2 \rangle \ll A_0^2, \tag{9.42}$$

so that $A - A_0 \ll A_0$ with high probability. Therefore it is a good approximation to replace A by A_0 in the indicated equation, which gives the following equation satisfied by the phase:

$$\dot{\varphi} = \Delta - \Delta_s \sin \varphi + \frac{\xi_2}{A_0}. \tag{9.43}$$

This equation can be used if the conditions (9.37) and (9.40) hold, i.e., if linearization in amplitude is allowed but not in phase. The Fokker-Planck equation corresponding to (9.43) is

$$\dot{w}(\varphi) = -\frac{\partial}{\partial \varphi}[(\Delta - \Delta_s \sin \varphi) w(\varphi)] + \frac{K}{2A_0^2} \frac{\partial^2 w(\varphi)}{\partial \varphi^2}, \tag{9.44}$$

and expresses the law of "conservation of probability"

$$\dot{w} + \frac{\partial G}{\partial \varphi} = 0,$$

where

$$G(\varphi) = (\Delta - \Delta_s \sin \varphi) w(\varphi) - \frac{K}{2A_0^2} \frac{\partial w(\varphi)}{\partial \varphi} \tag{9.45}$$

is just the "probability current" crossing the abscissa φ. Introducing the notation

$$D = \frac{2A_0^2 \Delta}{K}, \quad D_s = \frac{2A_0^2 \Delta_s}{K} = \frac{EA_0\omega_s}{K}, \tag{9.46}$$

we can write (9.45) in the form

$$G(\varphi) = \frac{K}{2A_0^2}\left[(D - D_s \sin \varphi) w - \frac{\partial w}{\partial \varphi}\right]. \tag{9.47}$$

We are interested in the stationary phase distribution, characterized by the fact that $w(\varphi)$ does not vary in time, or equivalently, that the probability current (9.47) is the same for every φ. In this case, the

function $w(\varphi)$ satisfies the equation

$$\frac{d^2w}{d\varphi^2} - \frac{d}{d\varphi}\left[(D - D_1 \sin\varphi)w\right] = 0. \tag{9.48}$$

The general solution of (9.48), containing two arbitrary constants C_1 and C_2, is given by

$$w(\varphi) = C_1\, e^{D\varphi + D_1\cos\varphi} \int_{C_2}^{\varphi} e^{-D\psi - D_1\cos\psi}\, d\psi. \tag{9.49}$$

To determine the constants C_1 and C_2 we have two conditions, the periodicity condition

$$w(\varphi + 2\pi) = w(\varphi) \tag{9.50}$$

and the normalization condition

$$\int_0^{2\pi} w(\varphi)\, d\varphi = 1. \tag{9.51}$$

To satisfy the first condition, we set $C_2 = \infty$, since then

$$w(\varphi + 2\pi) = C_1\, e^{2\pi D}\, e^{D\varphi + D_1\cos\varphi} \int_{\infty}^{\varphi + 2\pi} e^{-D\psi - D_1\cos\psi}\, d\psi$$

$$= C_1\, e^{2\pi D}\, e^{D\varphi + D_1\cos\varphi} \int_{\infty}^{\varphi} e^{-D(\psi' + 2\pi) - D_1\cos\psi'}\, d\psi' = w(\varphi).$$

Moreover,

$$w(\varphi) = C_1\, e^{2\pi D}\, e^{D\varphi + D_1\cos\varphi} \int_{\infty}^{\varphi + 2\pi} e^{-D\psi - D_1\cos\psi}\, d\psi$$

$$= C_1\, e^{2\pi D}\, e^{D\varphi + D_1\cos\varphi} \left\{ \int_{\varphi}^{\varphi + 2\pi} e^{-D\psi - D_1\cos\psi}\, d\psi \right.$$

$$\left. + \int_{\varphi - 2\pi}^{\varphi} e^{-D\psi - D_1\cos\psi}\, d\psi + \ldots \right\},$$

$$= C_1\, e^{2\pi D}\, e^{D\varphi + D_1\cos\varphi}(1 + e^{2\pi D} + \ldots) \int_{\varphi}^{\varphi + 2\pi} e^{-D\psi - D_1\cos\psi}\, d\psi,$$

i.e.,

$$w(\varphi) = \frac{1}{N} e^{D\varphi + D_s \cos\varphi} \int_{\varphi}^{\varphi + 2\pi} e^{-D\psi - D_s \cos\psi} d\psi, \tag{9.52}$$

where the normalization factor N is related to C_1 by the formula

$$N = -\frac{1 - e^{-2\pi D}}{C_1}.$$

Introducing the variable $\chi = \psi - \varphi$, we transform the normalization integral (9.51) into

$$N = \int_0^{2\pi} \int_0^{2\pi} \exp\left\{-D\chi + 2D_s \sin\frac{\chi}{2} \sin\left(\varphi + \frac{\chi}{2}\right)\right\} d\chi\, d\varphi$$
$$= 2\pi \int_0^{2\pi} e^{-D\chi} I_0\left(2D_s \sin\frac{\chi}{2}\right) d\chi.$$

Then, making the change of variable $y = \frac{1}{2}(\pi - \chi)$ for $0 < \chi < \pi$ and $y = \frac{1}{2}(\chi - \pi)$ for $\pi < \chi < 2\pi$, we obtain

$$N = 8\pi e^{-\pi D} \int_0^{\pi/2} \cosh(2Dy) I_0(2D_s \cos y) dy$$
$$= 4\pi^2 e^{-\pi D} I_{iD}(D_s) I_{-iD}(D_s), \tag{9.53}$$

where in the last step we use formula (4.453.1) from Ryshik and Gradstein's handbook.[2] If the mismatch vanishes ($D = 0$), the phase distribution (9.52) becomes

$$w(\varphi) = \frac{1}{2\pi I_0(D_s)} e^{D_s \cos\varphi}, \tag{9.54}$$

and has the symmetric form shown in Figure 23. If $D \neq 0$, the form of the curve $w(\varphi)$ can be found by numerical integration. Figures 24 and

[2] I. M. Ryshik and I. S. Gradstein, *op. cit.*, p. 239. $I_{iD}(z)$ is the Bessel function of imaginary argument and imaginary order.

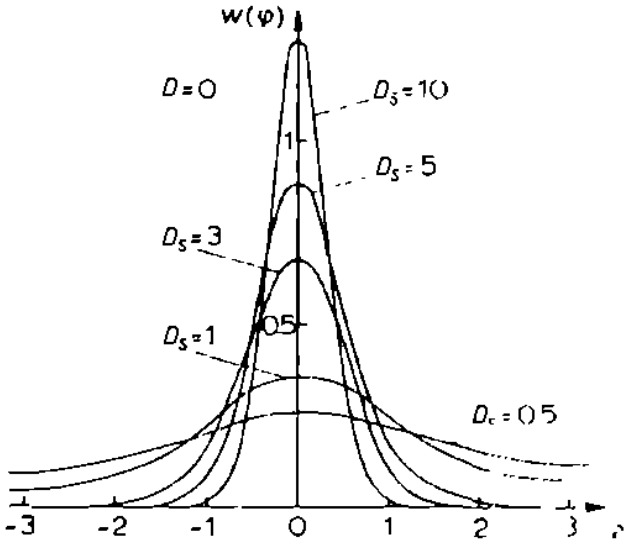

Fig. 23. The probability density of the phase for zero mismatch

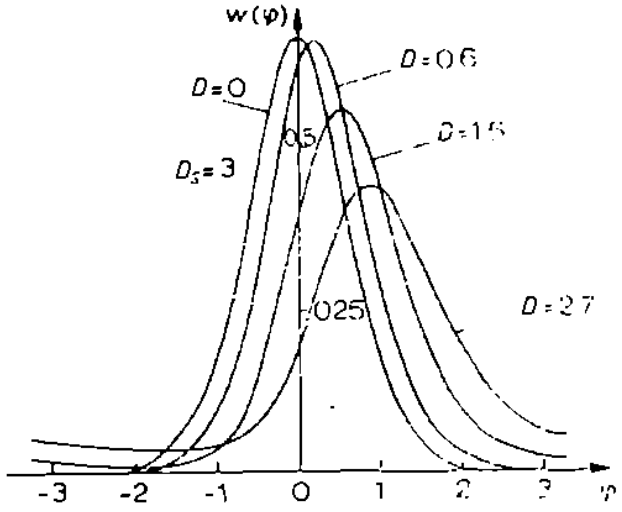

Fig. 24. The probability density of the phase for $D_s = 3$.

25 show the probability densities corresponding to the values $D_s = 3$ and $D_s = 10$ for various values of the parameter D.

The phase distribution (9.52) is essentially non-Gaussian because of the nonlinearity of the function $D - D_s \sin\varphi$. In fact, the motion of the representative point along the φ-axis can be interpreted as motion of a Brownian particle in an external force field whose dependence on the

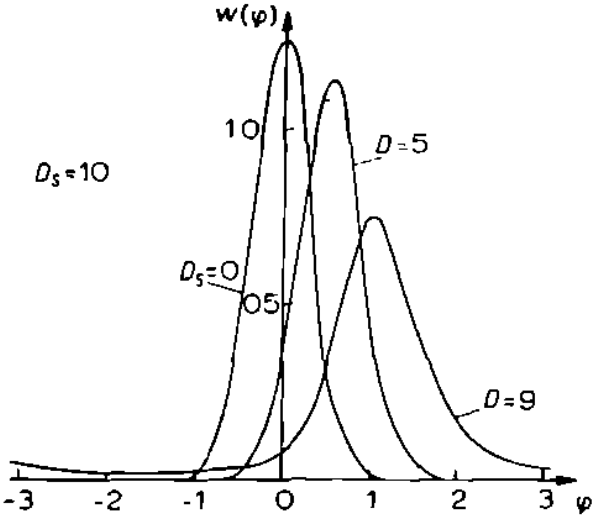

Fig. 25. The probability density of the phase for $D_s = 10$.

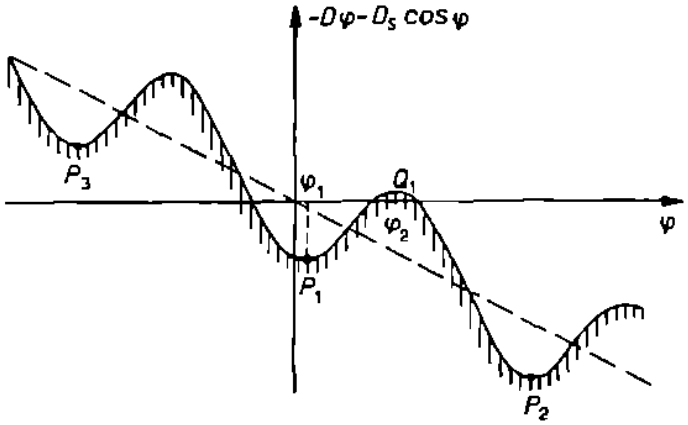

Fig. 26. Phase fluctuations as the random motion of a mass point along a corrugated inclined plane.

position is nonlinear, i.e., proportional to $D - D_s \sin\varphi$. The external forces can be derived from the potential

$$-D\varphi - D_s \cos\varphi,$$

which is the same as the potential of a mass point (particle) moving along a corrugated inclined plane (see Figure 26). In the absence of fluctuations, the representative point comes to rest at the bottom P_1 of one of the "wells." However, it should be noted that a synchronous regime is impossible if there are "no bottoms," because of excessive slope of

the plane, since then the representative point will "slide down the plane" without coming to rest.) The presence of random impacts causes occasional motion along the plane even when the representative point is "bound in a well." The effect of the random impacts is two-fold: While "knocking the particle around the bottom of a well," they manage now and then to "kick the particle out of the well," causing it to slide down to the next lowest well P_2, or less often to "land in a higher well P_3." In the first case, the phase increases by 2π, and in the second case it decreases by 2π. The net effect is to cause "sliding" of the representative point down the inclined plane. The sliding occurs more rapidly the greater the slope (mismatch), the shallower the wells (i.e., the smaller the synchronizing amplitude), and the stronger the random excitation. In other words, there is a regular flow of probability current down the slope, causing the frequency of the oscillations to shift in the direction of the fundamental frequency of the oscillator.

To find the mean frequency, we average equation (9.43), obtaining

$$\langle\dot{\varphi}\rangle = \langle\Delta - \Delta_s \sin\varphi\rangle = \int_0^{2\pi} (\Delta - \Delta_s \sin\varphi) w(\varphi)\, d\varphi .$$

According to (9.45), the integrand equals

$$G + \frac{K}{2A_0^2}\frac{dw}{d\varphi},$$

and hence,

$$\langle\dot{\varphi}\rangle = 2\pi G, \quad \text{i.e.,} \quad \langle\omega\rangle - \omega_s = 2\pi G,$$

since G is constant and w is periodic. It follows that

$$\langle\omega\rangle - \omega_s = \pi\frac{K}{A_0^2}\frac{1 - e^{-2\pi D}}{N} = \Delta\frac{\sinh \pi D}{\pi D}\,|I_{iD}(D_s)|^{-2}, \quad (9.55)$$

where we have used (9.52) to calculate the probability current (9.47). We now examine the results which can be deduced from this formula.

In Figure 27 we plot several curves showing the dependence of the frequency shift $\langle\omega\rangle - \omega_s$ on the relative mismatch Δ/Δ_s for various values of the parameter D_s. For the value $D_s = 0$, corresponding to very

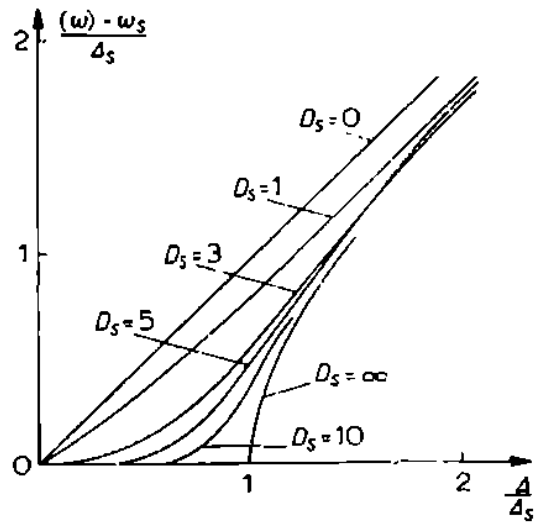

Fig. 27. Dependence of the frequency shift on the mismatch at various noise levels.

strong noise, (9.55) reduces to

$$\langle\omega\rangle - \omega_s = \Delta, \quad \text{i.e.,} \quad \langle\omega\rangle = \omega_0.$$

Thus the synchronizing signal has no effect if it is completely immersed in the noise, and the frequency of the oscillations is just the fundamental frequency of the oscillator. In the opposite extreme case, where the noise is very weak, we have[3]

$$\langle\omega\rangle - \omega_s \approx \begin{cases} 0 & \text{for} \quad \Delta < \Delta_s, \\ \sqrt{\Delta^2 - \Delta_s^2} & \text{for} \quad \Delta > \Delta_s. \end{cases}$$

[3] In fact, neglecting the noise term in (9.43), we obtain

$$\dot{\varphi} = \Delta - \Delta_s \sin\varphi,$$

and hence a synchronous regime $\dot{\varphi} = 0$ (i.e., $\langle\omega\rangle - \omega_s = 0$) is possible if $\Delta < \Delta_s$ but not if $\Delta > \Delta_s$. In the latter case, integration of the equation for $\dot{\varphi}$ gives the following expression for the time it takes a phase increment $\Delta\varphi = 2\pi$ to accumulate:

$$\Delta t = \int_{-\pi}^{\pi} \frac{d\varphi}{\Delta - \Delta_s \sin\varphi} = \frac{2\pi}{\sqrt{\Delta^2 - \Delta_s^2}}.$$

But then

$$\langle\dot{\varphi}\rangle = \langle\omega\rangle - \omega_s = \frac{\Delta\varphi}{\Delta t} = \sqrt{\Delta^2 - \Delta_s^2}.$$

The remaining curves, constructed by using tables of Bessel functions of imaginary argument and imaginary order, all lie between the indicated limiting curves. Correspondingly, the mean frequency of the oscillations lies between the fundamental frequency of the oscillator and the frequency of the external synchronizing signal.

In the case of greatest practical interest, the condition $D_s \gg 1$ is satisfied, since only then are regimes close to the synchronous regime possible. Here, depending on the relation between Δ and Δ_s, we can derive a number of formulas that are simpler than (9.55). If $\Delta \ll \Delta_s$, we can use the asymptotic formula for I_p, which together with (9.55) gives

$$\langle\omega\rangle - \omega_s = 2\Delta_s e^{-2D_s} \sinh \pi D. \tag{9.56}$$

If Δ is of order Δ_s or larger, then $D \gg 1$ and we can use the asymptotic expansion

$$I_{iD}(D_s) = \frac{v}{2\sqrt{3}} e^{\pi D/2}\left[e^{-\varrho + \pi i/6} H_{1/3}^{(1)}\left(\frac{i}{3} Dv^3\right) + e^{\varrho - \pi i/6} H_{1/3}^{(2)}\left(\frac{i}{3} Dv^3\right)\right],$$
$$D > 0, \quad Dv = \sqrt{D_s^2 - D^2} > 0, \tag{9.57}$$
$$\varrho = D\left(v - \frac{v^3}{4} - \arctan v\right).$$

suitable for large imaginary values of the order $p = iD$, with an error of absolute value less than $24\sqrt{2}/|p|$.[4]

Formula (9.57) leads to different simplified expressions, depending on whether or not the mismatch Δ is near the synchronization bandwidth Δ_s, or rather on the size of the ratio between $(\Delta_s/\Delta)^2 - 1$ and $D^{-2/3}$. Suppose that Δ is not too near Δ_s, or more exactly that $(\Delta_s/\Delta)^2 - 1 \gg D^{-2/3}$, so that $v^2 D^{2/3} \gg 1$. Then the Hankel functions in (9.57) have arguments much larger than unity, and hence can be replaced by their asymptotic representations. Therefore (9.55) gives

$$\langle\omega\rangle - \omega_s = \Delta v \exp\{-2D(v - \arctan v)\},$$

i.e.,

$$\langle\omega\rangle - \omega_s = \sqrt{\Delta_s^2 - \Delta^2} \exp\left\{-2\left(\sqrt{D_s^2 - D^2} - D \arccos \frac{D}{D_s}\right)\right\}. \tag{9.58}$$

[4] I. M. Ryshik and I. S. Gradstein, *op. cit.*, formula (6.464), p. 321.

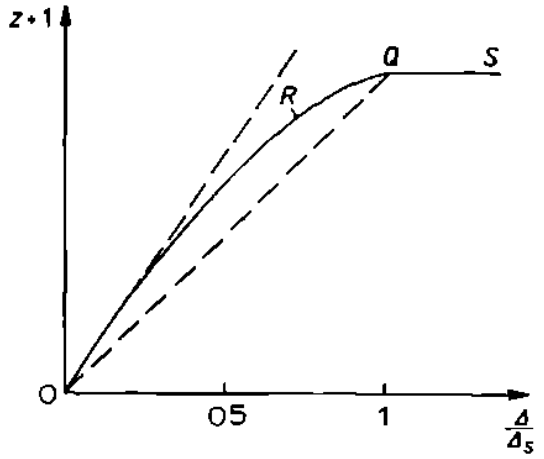

Fig. 28. Dependence of the mean frequency on the mismatch for small noise intensities.

The relation (9.58) can be written in the form

$$z \equiv \frac{1}{2D_s} \ln \left| \frac{\langle \omega \rangle - \omega_s}{\sqrt{\Delta_s^2 - \Delta^2}} \right| = f\left(\frac{\Delta}{\Delta_s}\right) - 1, \tag{9.59}$$

and is plotted in Figure 28. Here f is the function

$$f(u) = u \arccos u - \sqrt{1 - u^2} + 1 \quad (u \leqslant 1),$$

and has a graph lying between the lines u and $1.57u$ [for $u \geqslant 1$, we plot the function $f(u) = 1$]. Formulas (9.58) and (9.59) show the rapid growth of the number of times synchronization is lost as the mismatch or the noise level is increased.

Next we examine more closely the range where Δ/Δ_s is of order unity, i.e., the range where the synchronous regime is destroyed because the mismatch approaches the boundary of the synchronization band. If

$$\frac{\Delta_s}{\Delta} - 1 \sim \frac{D^{-2/3}}{2},$$

then the argument Dv^3 in (9.57) is of order unity, and hence we must take account of the detailed behavior of the Hankel functions, transforming them into other cylinder functions of order $\pm\frac{1}{3}$. In this case, we hvae $v \ll 1$, a fact which allows us to neglect the quantity $\varrho \approx -\frac{1}{5}(Dv^3)v^2$,

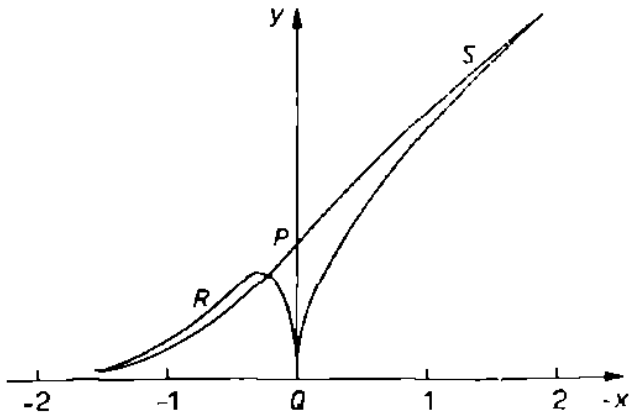

Fig. 29. Dependence of the mean frequency on the mismatch near the boundary of the synchronization band.

since $|\varrho| \ll 1$. Writing $\frac{1}{3}Dv^3 = x^{3/2}$ and using (9.55) and (9.57), we obtain[5]

$$y = \frac{3}{2\pi x}\left[I^2_{-1/3}(x^{3/2}) + I_{-1/3}(x^{3/2})I_{1/3}(x^{3/2}) + I^2_{1/3}(x^{3/2})\right]^{-1}, \quad (9.60)$$

where

$$y = \left(\frac{D}{3}\right)^{1/3}\frac{\langle\omega\rangle - \omega_s}{\Delta}, \quad x \approx 2\left(\frac{D}{3}\right)^{2/3}\left(\frac{\Delta_s}{\Delta} - 1\right) \approx 2\left(\frac{D_s}{3}\right)^{2/3}\left(1 - \frac{\Delta}{\Delta_s}\right). \quad (9.61)$$

This dependence is shown by the curve RP in Figure 29. Starting from about $x = 0.7$, the curve RP is close to the curve RQ representing the asymptotic expression

$$y = x^{1/2}\exp\left(-2x^{3/2}\right).$$

When the mismatch coincides with the synchronization bandwidth ($x = 0$), formula (9.60) gives

$$y = \frac{3}{2\pi}2^{-2/3}\Gamma^2\left(\tfrac{2}{3}\right) \approx 0.55,$$

$$\langle\omega\rangle - \omega_s \approx 0.55\,\Delta_s\left(\frac{3}{D_s}\right)^{1/3} = 0.55\,\frac{\omega_s}{2}\left[\tfrac{3}{2}E^2A_0^{-4}\omega_s\varkappa_\eta(\omega_s)\right]^{1/3}. \quad (9.62)$$

[5] Cf. I. M. Ryshik and I. S. Gradstein, *op. cit.*, formulas (6.493.1) and (6.493.2), p. 328.

As the mismatch is increased and passes through the boundary of the synchronization band, the frequency shift continues to grow. Then $\Delta > \Delta_s$, $x < 0$ and (9.55), (9.57) imply[6]

$$y = -\frac{2}{\pi x}\left[J_{1/3}^2(|x|^{3/2}) + N_{1/3}^2(|x|^{3/2})\right]^{-1}, \tag{9.63}$$

plotted in Figure 29 as the curve PS. As can be seen from the figure, starting from about $-x = 1$, the curve PS is close to the curve QS representing the asymptotic expression

$$y = (-x)^{1/2}, \quad \text{i.e.,} \quad \langle\omega\rangle - \omega_s = \sqrt{\Delta^2 - \Delta_s^2}.$$

The curves RQ and QS in Figure 29 represent the same functional dependence (in different variables) as the curves RQ ($z = f - 1$) and QS ($z = 0$) in Figure 28. Hence the curve $ORQS$ shown in Figure 28 is essentially accurate except in the region

$$\left|\frac{\Delta}{\Delta_s} - 1\right| \lesssim 0.5\left(\frac{3}{D_s}\right)^{2/3},$$

where the curve RQS should be replaced by the smoothed monotonic curve RPS shown in Figure 29. Thus the theory presented above allows us to follow continuously the transition from the regime of "deep synchronization" at small noise levels (where the linearization method

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[6] According to I. M. Ryshik and I. S. Gradstein, *op. cit.*, formulas (6.486.6) and (6.486.7), p. 327, for $x = -|x| = e^{\pi i}|x|$, we have

$$H_{1/3}^{(1)}(ix^{3/2}) = H_{1/3}^{(1)}(e^{2\pi i}|x|^{3/2}) = e^{-2\pi i/3} H_{1/3}^{(1)}(|x|^{3/2}) - 2e^{-\pi i/3} J_{1/3}(|x|^{3/2}),$$

$$H_{1/3}^{(2)}(ix^{3/2}) = H_{1/3}^{(2)}(e^{2\pi i}|x|^{3/2}) = e^{-2\pi i/3} H_{1/3}^{(2)}(|x|^{3/2}) + 2e^{\pi i/3} J_{1/3}(|x|^{3/2}).$$

It follows that

$$e^{\pi i/6} H_{1/3}^{(1)}(ix^{3/2}) + e^{-\pi i/6} H_{1/3}^{(2)}(ix^{3/2}) = (i + e^{-5\pi i/6}) H_{1/3}^{(2)}(|x|^{3/2}),$$

and hence

$$|e^{\pi i/6} H_{1/3}^{(1)}(ix^{3/2}) + e^{-\pi i/6} H_{1/3}^{(2)}(ix^{3/2})|^2 = |i - e^{\pi i/6}|^2 |H_{1/3}^{(2)}(|x|^{3/2})|^2$$
$$= J_{1/3}^2(|x|^{3/2}) + N_{1/3}^2(|x|^{3/2}).$$

is applicable) to the nonsynchronous regime that results when the mismatch approaches the synchronization band and passes through it. In the same way, we can study how synchronization is lost as the amplitude of the external synchronizing signal is decreased. In this case, for a given mismatch, the synchronization band shrinks until finally the mismatch lies outside the band. In any event, as the synchronous regime is destroyed, the frequency of the self-excited oscillations gradually changes from the synchronizing frequency ω_1 to the fundamental frequency ω_0 of the oscillator.

3. Large Phase Deviations and Diffusion of the Number of Oscillations

There are a number of other problems (especially those involving the time rate of change of the phase), where more is needed than the stationary solution (9.52) of the Fokker-Planck equation. We now consider the approximate solution of a number of nonstationary problems (like the one just mentioned) connected with first-passage times.

If there is no noise and if the synchronization condition $\Delta < \Delta_s$ holds, then, according to (9.43), the phase takes the stable synchronous value

$$\varphi_1 = \arcsin \frac{\Delta}{\Delta_s}.$$

The presence of the random impacts $\xi_2(t)$ causes fluctuational deviations from this synchronous phase. Suppose we ask how often the phase deviations $\varphi - \varphi_1 = \delta$ acquire values greater than a given value b, i.e., how often the graph of the function $\varphi(t) - \varphi_1 = \delta\varphi(t)$ reaches a given level b. In the approximation of a delta-correlated process $\xi_2(t)$, the total number of intersections of the level b will be infinite, since each intersection is accompanied by a large (theoretically infinite) number of other intersections, due to fluctuational "jitter" of the phase. However, if the level b is much greater than the r.m.s. deviation $\sigma(\delta\varphi)$, the intersections occur in clusters, as described in Chap. 1, Sec. 2. After completing such a cluster, the representative point enters the

region of small deviations, where it spends a considerable amount of time before it returns to the level b, suddenly producing another large cluster of intersections.

The problem before us is to calculate the number of such clusters which occur per unit time. This will tell us how often the phase leaves the region of the usual small deviations and attains the level b. In other words, we want to calculate the amount of time the representative point, with a small scatter near the synchronous value, takes to arrive at the level b. As we know, in the stochastic approximation, such problems involving first-passage times, are solved by using the nonstationary Fokker-Planck equation together with a zero boundary condition.

To avoid difficulties associated with finding the nonstationary solution of equation (9.44), we use the condition

$$\langle(\delta\varphi)^2\rangle \ll (\tfrac{1}{2}\pi - \varphi_1)^2. \tag{9.64}$$

Here (if $\Delta > 0$) $\frac{1}{2}\pi - \varphi_1$ is half the distance between the "well" P_1 in Figure 26 and the nearest "vertex" Q_1. The inequality (9.64) excludes the possibility of frequent interruptions of synchronization. It will be recalled that the grouping of intersections (or peaks) into clusters and the counting of these clusters makes sense only if $\langle(\delta\varphi)^2\rangle \ll b^2$, an inequality which coincides with (9.64) if $b \sim \frac{1}{2}\pi - \varphi_1$.

The condition (9.64) guarantees the applicability of the linearization method for investigating small deviations $\delta\varphi$ from the synchronous value of the phase. However, we are now interested in large deviations, and hence the problem is essentially nonlinear. Recalling the results found in the preceding section, we see that the case (9.64) can be characterized as the case where the relation (9.58) holds. In fact, it follows from (9.22), (9.24) and (9.37) that

$$\langle(\delta\varphi)^2\rangle \sim \frac{1}{2\delta'}\frac{K}{A_1^2} \sim \frac{1}{D_s}. \tag{9.65}$$

But if $\frac{1}{2}\pi - \varphi_1 \sim 1$, say, then

$$1 - \frac{\Delta^2}{\Delta_s^2} \sim 1,$$

and hence (9.64) implies the inequalities

$$D_s \gg 1, \quad \frac{\Delta_s^2}{\Delta^2} - 1 \gg D_s^{-2/3}\left(\frac{\Delta_s}{\Delta}\right)^{2/3} = D^{-2/3},$$

gnaranteeing the validity of (9.58).

To calculate the repetition frequency of the peak clusters, we use the method of Chap. 1, Sec. 2. Thus the following considerations have already been encountered, in large part. Suppose the first intersection in a cluster occurs at time t. After a time interval Δt somewhat exceeding the phase correlation time, the phase returns to the region of small deviations, and its conditional distribution has a form close to that of the stationary distribution. If the level b is raised sufficiently high above the r.m.s. deviation $\sigma(\delta\varphi)$ then the mean distance between clusters greatly exceeds Δt, and hence the mean time between first intersections in consecutive clusters can be regarded as the mean time it takes to go from a distribution near the stationary distribution to a first intersection of the level b.

To find this time, we consider the Fokker-Planck equation

$$\dot{w} = -\frac{\partial G}{\partial \varphi}, \tag{9.66}$$

which is the same as (9.44). But now the normalization condition (9.51) is satisfied only at the initial time, and instead, we solve (9.66) with the boundary condition

$$w(\varphi_b) = 0 \quad (\varphi_b = \varphi_1 + b). \tag{9.67}$$

According to (9.67), the representative point is excluded from consideration as soon as it reaches the level b, which "absorbs" the point. The number of points remaining under consideration is given by

$$W(t) = \int_{\varphi_b - 2\pi}^{\varphi_b} w\, d\varphi, \tag{9.68}$$

and slowly decreases at a rate determined by integrating (9.66):

$$\dot{W}(t) = -G(\varphi_b) + G(\varphi_b + 2\pi). \tag{9.69}$$

Here $G(\varphi_b)$ is the number of points absorbed per unit time at the level

b. Since we are not interested in the repetition rate of negative deviations, we must assume that the representative points are reflected rather than absorbed at the negative level $\varphi=\varphi_b-2\pi$, so that $G(\varphi_b-2\pi)=0$. Then (9.69) becomes

$$\dot{W}(t) = -G(\varphi_b). \tag{9.70}$$

The number of points reaching the boundary in the time interval between t and $t+dt$, i.e., taking time t to arrive at the boundary, equals $-dW=G(\varphi_b)dt$. Therefore the mean arrival time is given by

$$\langle t\rangle = -\int_0^\infty t\,dW(t) = \int_0^\infty tG(\varphi_b,t)\,dt. \tag{9.71}$$

As we now show, the number of remaining representative points has a decay law which is approximately exponential. Because of the condition $\langle(\delta\varphi)^2\rangle \ll b^2$, the form of the distribution is practically time-invariant, and only the "amplitude" of the distribution changes:

$$w(\varphi,t) = w_1(\varphi)v(t) \qquad \left(\int w_1\,d\varphi = 1\right). \tag{9.72}$$

Substituting (9.72) into (9.66), we obtain

$$\begin{aligned} \frac{\dot{v}}{v} &= -\frac{1}{w_1}\frac{\partial G_1}{\partial\varphi} = -\Lambda, \\ G_1 &= (\Delta - \Delta_s\sin\varphi)\,w_1 - \frac{K}{2A_0^2}\frac{\partial w_1}{\partial\varphi}, \end{aligned} \tag{9.73}$$

where Λ can only be a constant. This gives the equation $\dot{v}+\Lambda v=0$, whose solution is an exponential, and then (9.72) becomes

$$w(\varphi,t) = w_1(\varphi)e^{-\Lambda t}. \tag{9.74}$$

Substituting (9.74) into (9.71), we find that the constant Λ is the reciprocal of the mean arrival time:

$$\Lambda = \langle t\rangle^{-1}. \tag{9.75}$$

In other words, Λ is just the desired repetition rate of first intersections.

Taking account of (9.68), (9.70) and the factorization (9.72), we have

$$\Lambda = \frac{1}{W} G(\varphi_b) = G_1(\varphi_b), \tag{9.76}$$

where (9.73) or (9.47) can be used to express the current $G_1(\varphi_b)$ in terms of the probability density $w_1(\varphi)$. We now examine in more detail the distribution $w_1(\varphi)$, corresponding to the regime of "stable decay" of the "unabsorbed" representative points. Such a point spends most of its time in the region $\delta\varphi \sim \sigma(\delta\varphi)$, far from the absorbing level b, and hence its motion differs only slightly from the motion under stationary conditions. This means that the form of the curve $w_1(\varphi)$ for $\delta\varphi \sim \sigma(\delta\varphi)$ is very close to the stationary distribution (9.52). However, near the absorbing boundary, the density $w_1(\varphi)$ differs appreciably from (9.52); in fact, it has smaller values but a steeper gradient, corresponding to increased flow in the direction of the absorbing boundary.

Because of the underlying condition $\sigma(\delta\varphi) \ll b$, the quantity Λ is small, and hence it can be assumed that fluctuations of all sizes $[\sigma(\delta\varphi) \lesssim \delta\varphi \lesssim b]$ are described by the original equation (9.48):

$$\frac{d^2 w_1}{d\varphi^2} - \frac{d}{d\varphi}[(D - D_s \sin\varphi) w_1] = 0. \tag{9.77}$$

The normalization condition accompanying this equation is the same as before, but the periodicity condition (9.50) is replaced by the "absorption condition"

$$w_1(\varphi_b) = 0. \tag{9.78}$$

The solution of (9.77) with the boundary condition (9.78) is

$$w_1(\varphi) = \frac{1}{N_1} e^{D\varphi + D_s \cos\varphi} \int_{\varphi}^{\varphi_b} e^{-D\psi - D_s \cos\psi} d\psi, \tag{9.79}$$

obtained from (9.49) by setting

$$C_1 = -\frac{1}{N_1}, \quad C_2 = \varphi_b.$$

In the region $\delta\varphi \sim \sigma(\delta\varphi)$, where the expression $D\varphi + D_s\cos\varphi$ has its maximum, the integrals in (9.52) and (9.79) differ only slightly from

the expression

$$w_2 = \frac{1}{N_2} e^{D\varphi + D_s \cos \varphi}, \tag{9.80}$$

and hence differ only slightly from each other, as anticipated above. But in this region we can use the expansion

$$\begin{gathered} D\varphi + D_s \cos \varphi \approx D\varphi_1 + D_s \cos \varphi_1 - \tfrac{1}{2}\sigma_\varphi^{-2}(\delta\varphi)^2 \\ (\sigma_\varphi^{-2} = D_s \cos \varphi_1 = \sqrt{D_s^2 - D^2}), \end{gathered} \tag{9.81}$$

allowing us to transform (9.80) into the form

$$w_2 = \frac{1}{N_2} e^{D\varphi_1 + D_s \cos \varphi_1 - \frac{1}{2}\sqrt{D_s^2 - D^2}(\delta\varphi)^2} = \frac{1}{\sqrt{2\pi}\sigma_\varphi} e^{-(\delta\varphi)^2/2\sigma_\varphi^2}. \tag{9.82}$$

Using (9.82) to evaluate N_2, and comparing (9.79) and (9.80), we find the normalization factor in (9.79):

$$N_1 = N_2 \int_{\varphi_1}^{\varphi_b} e^{-D\psi - D_s \cos \psi}\, d\psi = \sqrt{2\pi}\sigma_\varphi\, e^{D\varphi_1 + D_s \cos \varphi_1} \int_{\varphi_1}^{\varphi_b} e^{-D\psi - D_s \cos \psi}\, d\psi . \tag{9.83}$$

Then, to find the frequency (9.76) of the peak clusters, we need only substitute the density (9.79) into the expression (9.47) for the current $G_1(\varphi_b)$, obtaining

$$\Lambda = \frac{K}{2A_0^2 N_1}. \tag{9.84}$$

The integral appearing in (9.83) can be evaluated approximately, by using the condition $D_s \gg 1$, which implies that the integrand is appreciably different from zero only in a narrow band near φ_b. In fact, after using the expansion

$$\begin{aligned} D\varphi + D_s \cos \varphi \approx D\varphi_b + D_s \cos \varphi_b + (D - D_s \sin \varphi_b)(\varphi - \varphi_b) \\ - \tfrac{1}{2}D_s \cos \varphi_b (\varphi - \varphi_b)^2, \end{aligned} \tag{9.85}$$

(9.83) and (9.84) give

$$\Lambda^{-1} \approx \frac{2A_0^2}{K}\sqrt{2\pi}\sigma_\varphi\, e^{\delta u} \int_{-\infty}^{0} \exp\{(D_s \sin \varphi_b - D)\varrho - \tfrac{1}{2}D_s |\cos \varphi_b| \varrho^2\}\, d\varrho \tag{9.86}$$

$(\cos\varphi_b<0)$, where

$$\delta u = D\varphi_1 + D_s\cos\varphi_1 - D\varphi_b - D_s\cos\varphi_b$$

is a kind of "potential difference" between the points φ_1 and φ_b, relative to the energy of the noise. Here we assume that the exponential in the integrand of (9.86) falls off so rapidly that it becomes vanishingly small for $\varrho=\varphi_1-\varphi_b$ (this is the reason for extending the lower limit of integration to $-\infty$). It should be noted that such a rapid decay is actually implied by $\sigma_\varphi \ll b$. In terms of the complementary error function

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}\,dy,$$

we can write (9.86) in the form

$$\Lambda^{-1} = \frac{\pi}{\Delta_s}(\cos\varphi_1\,|\cos\varphi_b|)^{-1/2}\exp\left\{\delta u + \frac{D_s(\sin\varphi_b-\alpha)^2}{2|\cos\varphi_b|}\right\} \times \operatorname{erfc}\left\{(\sin\varphi_b-\alpha)\left(\frac{D_s}{2|\cos\varphi_b|}\right)^{1/2}\right\}, \tag{9.87}$$

where $\alpha=D/D_s$.

The most inaccessible value of the phase

$$\varphi_2 = \pi - \varphi_1 = \pi - \arcsin\frac{D}{D_s} \tag{9.88}$$

corresponds to the "vertex" Q_1 of the potential function (see Figure 26), and hence the maximum potential difference is

$$\delta u = 2D_s\cos\varphi_1 - D(\varphi_2-\varphi_1) = 2\sqrt{D_s^2-D^2} - 2D\arccos\frac{D}{D_s}. \tag{9.89}$$

At the point Q_1, the potential function has the same second derivative as the "lowest point" P_1 (where $\varphi=\varphi_1$), but with the opposite sign $(D_s\cos\varphi_2=-\sigma_\varphi^2)$. Setting $\varphi_b=\varphi_2$ in (9.87), we find that the phase, which usually stays near φ_1, occasionally grows and reaches the

"vertex" Q_1, with frequency

$$A = \frac{\Delta_s}{\pi} e^{-\delta u} \cos \varphi_1$$

$$= \frac{1}{\pi} \sqrt{\Delta_s^2 - \Delta^2} \exp \left\{ -2 \left(\sqrt{D_s^2 - D^2} - D \arccos \frac{D}{D_s} \right) \right\}. \quad (9.90)$$

Once having arrived at the vertex Q_1, the representative point can slide down either side of the "hill" with equal probability. Thus, in half the cases, the point will slide back to the original well P_1. On the other hand, in the remaining cases, the point will "get over the top" and slide down to a new well P_2, lying further downhill, and as a result the phase will acquire an increment of 2π. Obviously, the phase jumps in the forward direction occur with a frequency G_+ equal to one half of (9.90):

$$G_+ = \frac{1}{2\pi} \sqrt{\Delta_s^2 - \Delta^2} \exp \left\{ -2 \left(\sqrt{D_s^2 - D^2} - D \arccos \frac{D}{D_s} \right) \right\}. \quad (9.91)$$

Besides "forward" phase jumps, there are also "backward" phase jumps, which decrease the phase by 2π and occur with frequency G_-. To calculate G_-, we must choose a value of φ_b corresponding to the next vertex on the "uphill" side of φ_1. This value is $-\pi - \varphi_1$, and instead of (9.91) we get

$$G_- = \frac{1}{2\pi} \sqrt{\Delta_s^2 - \Delta^2} \exp \left\{ -2 \left(\sqrt{D_s^2 - D^2} + D \left(\pi - \arccos \frac{D}{D_s} \right) \right) \right\}$$

$$= e^{-2\pi D} G_+. \quad (9.92)$$

In the expressions (9.91) and (9.92), the determining factor is the exponential $e^{-\delta u}$, which resembles the Boltzmann distribution. The meaning of this factor is perfectly clear: The higher the barrier, the more difficult it is to overcome. Which of the vertices on the two sides of φ_1 is higher depends on the sign of the mismatch Δ. If $\Delta > 0$, then the inclination is in the direction shown in Figure 26, causing "forward" jumps to occur more frequently than "backward" jumps ($G_+ > G_-$). If $\Delta < 0$, we have the opposite inclination, and "backward" jumps predominate.

The difference between the expressions (9.91) and (9.92) gives the average uncompensated flow causing the mean frequency of the self-excited oscillations to shift towards the fundamental frequency of the oscillator, i.e.,

$$\langle\omega\rangle - \omega_s = 2\pi(G_+ - G_-) = \sqrt{A_s^2 - A^2}(1 - e^{-2\pi D}) \exp\left\{-2\left(\sqrt{D_s^2 - D^2} - D \arccos\frac{D}{D_s}\right)\right\}. \tag{9.93}$$

This formula implies (9.56) as a special case for $D \ll D_s$, and also (9.58) for $D \gg 1$. Interpreting $\langle\omega\rangle - \omega_s$ as the difference

$$2\pi(G_+ - G_-) = 2\pi G_+(1 - e^{-2\pi D}),$$

we observe that (9.55) is in the form of a difference between two terms differing by a factor of $e^{-2\pi D}$, just as in (9.91) and (9.92). Therefore it is natural to identify minuends and subtrahends separately, obtaining the formulas

$$\begin{aligned} G_+ &= \frac{K}{2A_0^2}\frac{1}{N} = \frac{1}{4\pi^2}\frac{K}{2A_0^2} e^{\pi D} |I_{iD}(D_s)|^{-2}, \\ G_- &= e^{-2\pi D} G_+ = \frac{1}{4\pi^2}\frac{K}{2A_0^2} e^{-\pi D} |I_{iD}(D_s)|^{-2} \end{aligned} \tag{9.94}$$

which generalize (9.91) and (9.92).

It follows from the above considerations that from time to time the phase jumps forward or backward by a period. The duration of each jump is of order

$$\frac{1}{A_s} \ln \frac{1}{\sigma_\varphi^2} \sim \frac{1}{A_s} \ln D_s, \tag{9.95}$$

while the average time between jumps is much larger, of order

$$\frac{1}{A_s} e^{\Delta u} \sim \frac{1}{A_s} e^{D_s}.$$

Therefore the jumps can be regarded as statistically independent. Examining the number n of extra or missing periods in the generated signal, as compared to the synchronizing signal, we have a model

reminiscent of a one-dimensional random walk, i.e., at random times the number n undergoes random jumps equal to either $+1$ or -1. Because of the operation of the central limit theorem, after a time t greatly exceeding the average interval between jumps, the random variable n approaches a Gaussian distribution, with mean and variance

$$\langle n\rangle = \frac{\langle\omega\rangle - \omega_s}{2\pi} t = (G_+ - G_-)\,t, \quad \mathbf{D}n = (G_+ + G_-)\,t \equiv D_0 t. \quad (9.96)$$

In fact, the number of positive jumps has mean $G_+ t$ and variance $G_+ t$, while the number of negative jumps has mean $G_- t$ and variance $G_- t$. Moreover, since the positive and negative jumps are independent, the means and variances add algebraically. The total number of oscillations after time t is obtained by increasing n by the nonrandom number of period $\omega_s t/2\pi$.

Thus the number of oscillations in the generated signal undergoes diffusional "spreading," with diffusion coefficient

$$D_0 = G_+ + G_- = \frac{1}{2\pi^2}\,\frac{K}{2A_0^2}\cosh \pi D\,|I_{iD}(D_s)|^{-2}. \quad (9.97)$$

The same is true of the phase, which is proportional to the number n and hence has a diffusion coefficient $(2\pi)^2$ times larger than that of n.

It follows from (9.93) and (9.94) that the diffusion coefficient (9.97) can be obtained by multiplying the frequency shift $(2\pi)^{-1}(\langle\omega\rangle - \omega_s)$ by $\coth \pi D$; if $D \gg 1$, this gives just $(2\pi)^{-1}(\langle\omega\rangle - \omega_s)$. Thus the behavior of the diffusion coefficient can be investigated by using the formulas and graphs of the preceding section.

Clearly, the diffusion of the number of oscillations increases with the noise intensity. For zero mismatch, (9.97) leads to the dependence

$$D_0 = \frac{1}{4\pi^2}\,\frac{K}{A_0^2 I_0^2(D_s)} = \frac{\Delta_s}{2\pi^2}\,\frac{1}{D_s I_0^2(D_s)}, \quad (9.98)$$

as shown in Figure 30.

In cases where the relations (9.91) and (9.92) hold, they can be

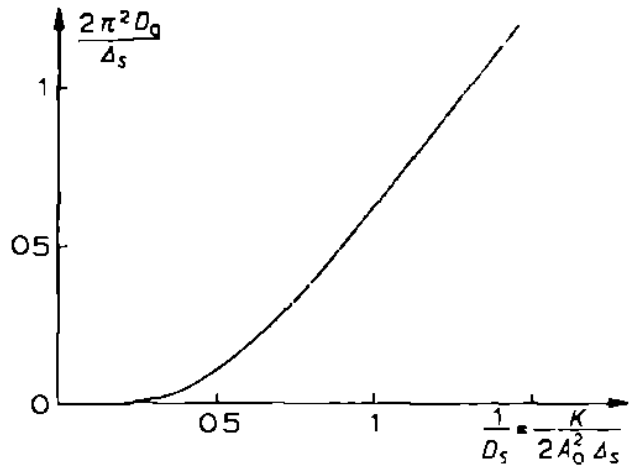

Fig. 30. Dependence of the diffusion coefficient on the noise intensity for zero mismatch.

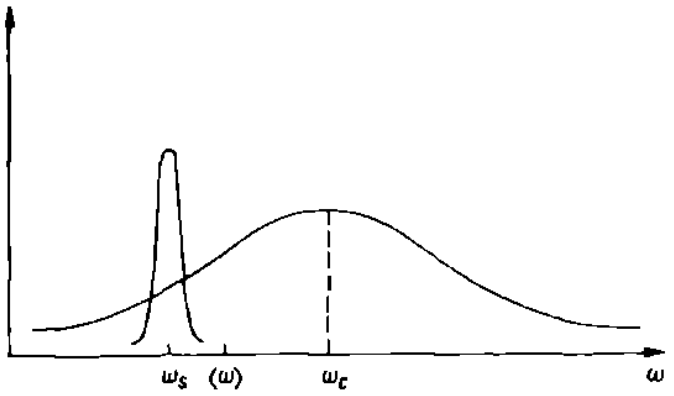

Fig. 31. Spectral density of the signal generated by a synchronized oscillator in the presence of noise ($\langle\omega\rangle - \omega_s \sim \Delta e^{-\delta u}$, $\omega_c - \omega_s \sim \Delta_s \tanh \pi D$).

added to give

$$D_0 = \frac{1}{2\pi}\sqrt{\Delta_s^2 - \Delta^2}\,(1 + e^{-2\pi D}) \times \exp\left\{-2\left(\sqrt{D_s^2 - D^2} - D \arccos\frac{D}{D_s}\right)\right\}. \quad (9.99)$$

As the mismatch is increased, the amount of diffusion increases monotonically and quite rapidly, as can be deduced from Figure 28, by writing z in the form

$$z = \frac{1}{2D_s} \ln \frac{2\pi D_0}{\sqrt{\Delta_s^2 - \Delta^2}}. \quad (9.100)$$

As for the spectral density of the generated signal, it must first of all contain a discrete spectral line of frequency ω_s due to the presence of synchronization. But $\langle\omega\rangle \neq \omega_s$, and hence the total spectral density has a continuous component, whose "center frequency" ω_c is shifted from the discrete line at ω_s by an amount of order $\Delta_s \tanh \pi D$, as shown schematically in Figure 31. The continuous component has bandwidth of order Δ_s and total power of order $A_0^2 e^{-\delta u}$, while the discrete component has power of order $A_0^2/2$.[7]

4. The Case of a Large Synchronizing Signal

The considerations of the preceding two sections were subject to the condition that the amplitude E of the external synchronizing signal be small, as expressed by the condition (9.39), but in practice E is often increased, in order to achieve a synchronous regime more quickly. Therefore it is of interest to study synchronization when the external amplitude E is comparable to εA_0. The small deviations from the synchronous regime corresponding to this case have already been considered in the linear approximation in Sec. 1. However, the nonlinear problem is very complicated, and we shall solve it by introducing a simplifying assumption of a different kind, i.e., by assuming that there is no mismatch ($\Delta = 0$). This case is of greater practical interest than the case of nonvanishing mismatch, since it is optimal in the sense of leading to the most stable synchronous regime.

It is now no longer appropriate to start from the equations (9.9). Instead we base our calculations on certain equations equivalent to (9.9), which can be derived as follows: Suppose we transform from polar coordinates A and φ to rectangular coordinates

$$z_1 = A\cos\varphi, \quad z_2 = A\sin\varphi \tag{9.101}$$

in a phase plane rotating with angular velocity ω_s. Differentiating

[7] The details are given in the Supplement, p. 273.

(9.101) with respect to time and using the equations (9.7), we find that

$$\dot{z}_1 = A\cos\varphi - A\dot{\varphi}\sin\varphi = \frac{\varepsilon\omega_0}{2}\left(1 - \frac{z_1^2 + z_2^2}{A_0^2}\right)z_1 + \frac{\omega_s E}{2} - \Delta z_2 + \omega_s\eta\sin\omega_s t,$$

$$\dot{z}_2 = A\sin\varphi + A\dot{\varphi}\cos\varphi = \frac{\varepsilon\omega_0}{2}\left(1 - \frac{z_1^2 + z_2^2}{A_0^2}\right)z_2 + \Delta z_1 + \omega_s\eta\cos\omega_s t. \quad (9.102)$$

The fluctuational terms $\omega_s\eta\sin\omega_s t, \omega_s\eta\cos\omega_s t$ appearing in (9.102) can be simplified much more easily than those appearing in (9.7), since the arguments of the trigonometric functions are independent of φ, and hence of η. If the correlation time of the process $\eta(t)$ is much smaller than the amplitude and phase relaxation times, then the correlation function

$$\langle\omega_s\eta\sin\omega_s t\cdot\omega_s\eta_\tau\sin\omega_s(t+\tau)\rangle = \frac{\omega_s^2}{2}\langle\eta\eta_\tau\rangle[\cos\omega_s\tau - \cos\omega_s(2t+\tau)]$$

can be replaced by

$$\delta(\tau)\int_{-\infty}^{\infty}\frac{\omega_s^2}{2}\langle\eta\eta_\tau\rangle\cos\omega_s\tau\,d\tau = K\delta(\tau),$$

and similarly for the second term $\omega_s\eta\cos\omega_s t$. As a result, we obtain the system

$$\begin{aligned}\dot{z}_1 &= \frac{\varepsilon\omega_0}{2}\left(1 - \frac{z_1^2 + z_2^2}{A_0^2}\right)z_1 + \frac{\omega_s E}{2} - \Delta z_2 + \xi_1,\\ \dot{z}_2 &= \frac{\varepsilon\omega_0}{2}\left(1 - \frac{z_1^2 + z_2^2}{A_0^2}\right)z_2 + \Delta z_1 + \xi_2,\end{aligned} \quad (9.103)$$

where ξ_1, ξ_2 are independent, identically distributed Gaussian random functions, with the properties (9.10), i.e., exactly the same as the functions ξ_1, ξ_2 in (9.9). There are other ways of proving the equivalence of the systems (9.9) and (9.103).

For $\Delta = 0$, the system (9.103) describes the motion of a Brownian particle in a potential force field. In fact, in this case (9.103) can be

written in the form

$$\dot{z}_1 = -\frac{\partial U}{\partial z_1} + \xi_1, \quad \dot{z}_2 = -\frac{\partial U}{\partial z_2} + \xi_2, \tag{9.104}$$

where

$$U(z_1, z_2) = \frac{\varepsilon\omega_0}{2}\left(\frac{z_1^2 + z_2^2}{2A_0^2} - 1\right)\frac{z_1^2 + z_2^2}{2} - \frac{\omega_s E}{2} z_1 \tag{9.105}$$

is the potential function shown schematically in Figure 32. The Fokker-Planck equation corresponding to the system (9.104) is

$$\dot{w} = \frac{\partial}{\partial z_1}\left(\frac{\partial U}{\partial z_1} w\right) + \frac{\partial}{\partial z_2}\left(\frac{\partial U}{\partial z_2} w\right) + \frac{K}{2}\left(\frac{\partial^2 w}{\partial z_1^2} + \frac{\partial^2 w}{\partial z_2^2}\right). \tag{9.106}$$

The corresponding stationary distribution must therefore satisfy the equation

$$\frac{\partial^2 w}{\partial z_1^2} + \frac{\partial^2 w}{\partial z_2^2} + \frac{2}{K}\frac{\partial}{\partial z_1}\left(\frac{\partial U}{\partial z_1} w\right) + \frac{2}{K}\frac{\partial}{\partial z_2}\left(\frac{\partial U}{\partial z_2} w\right) = 0, \tag{9.107}$$

with solution

$$w(z_1, z_2) = \frac{1}{N} e^{-2U/K}, \tag{9.108}$$

if we require, as usual, that the probability density w vanish at infinity. Just as in the one-dimensional case, (9.108) has the form of a Boltzmann distribution.

The stationary distribution (9.108) can easily be written in terms of the amplitude A and the phase φ. In fact, since $dz_1 dz_2 = A dA d\varphi$, it follows from (9.105) and (9.108) that

$$w(A, \varphi) = \frac{A}{N} \exp\left\{-\frac{\varepsilon\omega_0}{K}\left(\frac{A^2}{2A_0^2} - 1\right)\frac{A^2}{2} + \frac{\omega_s E}{K} A \cos\varphi\right\}. \tag{9.109}$$

To obtain the univariate amplitude or phase distribution, we need only integrate (9.109) with respect to the other variable. Thus, integrating (9.109) with respect to φ, we find the stationary amplitude distribution

$$w(A) = \frac{2\pi}{N} A \exp\left\{-\frac{\varepsilon\omega_0}{K}\left(\frac{A^2}{2A_0^2} - 1\right)\frac{A^2}{2}\right\} I_0\left(\frac{\omega_s E}{K} A\right),$$

where $I_0(z)$ is the modified Bessel function of the first kind, of order zero.

The case of greatest practical interest occurs when the noise is not strong enough to prevent anything resembling synchronization. Then the oscillator operates under essentially synchronous conditions, although occasionally the noise destroys the synchronous regime. The inequality

$$\frac{K}{\varepsilon\omega_0 A_0^2} \ll 1 \tag{9.110}$$

guarantees that the departures from the synchronous regime are not excessive, and the rest of our calculations will be based on the assumption that this condition holds. Then most of the probability (9.108) is concentrated near the synchronous values $z_1 = A_0 a_1$, $z_2 = 0$, and the standard deviations $\sigma(z_1), \sigma(z_2)$ are small enough to be treated by the linearization method, leading to the expressions (9.22). Under these conditions, we can use the expansion

$$U(z_1, z_2) \approx U_1 + \tfrac{1}{2}\delta(z_1 - A_0 a_1)^2 + \tfrac{1}{2}\delta' z_2^2,$$

where

$$\delta = \frac{\varepsilon\omega_0}{2}(3a_1^2 - 1), \quad \delta' = \frac{\varepsilon\omega_0}{2}(a_1^2 - 1). \tag{9.111}$$

For the usual small deviations, this leads to a Gaussian distribution

$$w(z_1, z_2) = \frac{1}{N} e^{-2U_1/K} \exp\left\{-\frac{\delta}{K}(z_1 - A_0 a_1)^2 - \frac{\delta'}{K} z_2^2\right\},$$

where it is a simple consequence of the normalization condition that

$$\frac{1}{N} e^{-2U_1/K} = \frac{1}{\pi K}\sqrt{\delta\delta'}. \tag{9.112}$$

As in Sec. 3, we now consider the nonlinear problem of determining the frequency of interruptions of synchronization, which lead either to the loss or the gain of a period. Here we are interested in the case where such interruptions occur rather seldom, i.e., where the condition (9.110) guaranteeing "satisfactory synchronization" holds, so that $\langle(\delta A)^2\rangle \ll A_0^2$, $\langle(\delta\varphi)^2\rangle \ll 1$. If Δ_s is not too large, the potential surface $y = U(z_1, z_2)$ corresponding to (9.105) is cup-shaped with a circular gutter at its bottom (see Figure 32), where the gutter is bounded by a loop around the origin and tilts downward in the

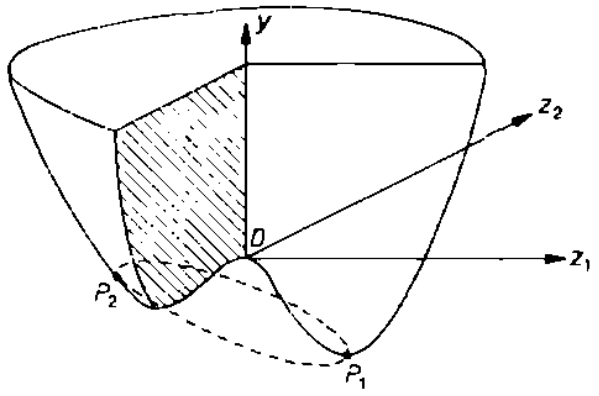

Fig. 32. The potential surface $y = U(z_1, z_2)$, shaped like a cup with an inclined circular gutter at the bottom.

direction of the z_1-axis. The lowest point of the gutter has coordinates $z_1 = A_0 a_1$, $z_2 = 0$, and corresponds to the synchronous values of the amplitude and phase. A phase jump occurs whenever the "fluctuational impacts" cause the representative point to move upward along the gutter until it reaches the highest point P_2 of (the bottom of) the gutter, afterwards sliding back into the original "well" from the opposite side. The highest point P_2 of the gutter has coordinates $z_1 = -A_0 a_2$, $z_2 = 0$, and lies on the opposite side of the origin from the lowest point (with coordinates $z_1 = A_0 a_1$, $z_2 = 0$).

It is reasonable to expect that the frequency of phase jumps will depend on the difference between the heights of the points P_1 and P_2, which we now determine by finding the coordinates of P_1 and P_2. Both points are extrema of the expression (9.105) with $z_2 = 0$. Thus, differentiating (9.105), we obtain the equation

$$a^3 - a - \frac{2\Delta_s}{\varepsilon\omega_0} = 0 \quad \left(a = \frac{z_1}{A_0},\ \Delta_s = \frac{\omega_s E}{2A_0}\right), \tag{9.113}$$

whose roots a_1 and a_2 determine the position of the points P_1 and P_2. The substitution $a = -(3p)^{-1/2} y$ reduces the cubic equation (9.113) to the standard form

$$y^3 - 3py + 2 = 0,$$

where

$$3p = \left(\frac{\varepsilon\omega_0}{\Delta_s}\right)^{2/3}$$

The roots of this equation, arranged in the order $y_1 > y_2 > 0 > y_3$ and tabulated as a function of $3p$, are available in the literature.[8] Therefore, given any value of the quantity $\varepsilon\omega_0/\Delta_s$, we can find the amplitudes

$$\begin{aligned} A_1 &= A_0 a_1, \quad a_1 = -\frac{y_3}{\sqrt{3p}} = |y_3| \sqrt[3]{\frac{\Delta_s}{\varepsilon\omega_0}}, \\ A_2 &= A_0 a_2, \quad a_2 = \frac{y_1}{\sqrt{3p}} = y_1 \sqrt[3]{\frac{\Delta_s}{\varepsilon\omega_0}}, \end{aligned} \tag{9.114}$$

corresponding to the points P_1 and P_2, as well as the values of the potential (9.105) at these points:

$$\begin{aligned} U_1 &= \frac{\varepsilon\omega_0}{2} A_0^2 \left[\frac{a_1^4}{4} - \frac{a_1^2}{2} - \frac{2\Delta_s}{\varepsilon\omega_0} a_1\right] \\ &= \frac{\varepsilon\omega_0 A_0^2}{2(3p)^2}\left[\left(\frac{y_3^2}{2}\right)^2 - 3p\frac{y_3^2}{2} + 2y_3\right] < 0, \\ U_2 &= \frac{\varepsilon\omega_0}{2} A_0^2 \left[\frac{a_2^4}{4} - \frac{a_2^2}{2} + \frac{2\Delta_s}{\varepsilon\omega_0} a_2\right] \\ &= \frac{\varepsilon\omega_0 A_0^2}{2(3p)^2}\left[\left(\frac{y_1^2}{2}\right)^2 - 3p\frac{y_1^2}{2} + 2y_1\right]. \end{aligned} \tag{9.115}$$

The problem of calculating the number of interruptions of synchronization is essentially that of calculating the number of times the representative point intersects the half-line or ray $z_1 < 0$ ($z_2 = 0$). This quantity is given by formula (9.76), except that now

$$\Lambda = G_1|_b$$

(b stands for "boundary") is the probability current arriving at the half-line $z_1 < 0$. Since here we are concerned with the two-dimensional case, the probability current corresponding to equation (9.106) is a vector with components

$$-\frac{\partial U}{\partial z_1} w - \frac{K}{2}\frac{\partial w}{\partial z_1}, \quad -\frac{\partial U}{\partial z_2} w - \frac{K}{2}\frac{\partial w}{\partial z_2}.$$

[8] See e.g., E. Jahnke and F. Emde, *Tables of Functions with Formulae and Curves*, fourth edition, Dover Publications, Inc., New York (1945), Addenda, p. 20 ff.

Therefore the current of representative points "absorbed" by the half-line $z_1 < 0$ from one side (from above, say) equals

$$G_1|_b = \frac{K}{2} \int_{-\infty}^{0} \left. \frac{\partial w_1}{\partial z_2} \right|_b dz_1, \tag{9.116}$$

since $w_1|_b = 0$.[9]

By analogy with Sec. 3, the probability density w_1 appearing in (9.116) differs only slightly from the stationary distribution (9.108) in the region where most of the probability accumulates. However, its behavior is quite different near the "absorbing" half-line, since it must obey the zero boundary condition

$$w_1 = 0 \quad \text{for} \quad z_1 < 0, \quad z_2 = 0. \tag{9.117}$$

Between the absorbing boundary and the region of maximum probability, the density $w_1(z_1, z_2)$ still satisfies (9.107). To find the current (9.116), we must find a solution, even if only approximate, of equation (9.107), subject to the boundary condition (9.117).

To this end, we look for a solution in the form

$$w_1(z_1, z_2) = v(z_1)\, w_2(z_1, z_2), \tag{9.118}$$

where the function

$$v(z_1) = \exp\left\{ -\frac{2}{K} U(z_1, 0) \right\} \tag{9.119}$$

satisfies the equation

$$\frac{\partial^2 v}{\partial z_1^2} + \frac{2}{K} \frac{\partial}{\partial z_1} \left(\frac{\partial U(z_1, 0)}{\partial z_1} v \right) = 0.$$

Substituting (9.118) into (9.107), we obtain the following exact equation for the function $w_2(z_1, z_2)$:

$$\frac{\partial^2 w_2}{\partial z_2^2} + \frac{2}{K} \frac{\partial}{\partial z_2} \left(\frac{\partial U}{\partial z_2} w_2 \right) = -\frac{\partial^2 w_2}{\partial z_1^2} - \left(\frac{2}{v} \frac{\partial v}{\partial z_1} + \frac{2}{K} \frac{\partial U}{\partial z_1} \right) \frac{\partial w_2}{\partial z_1} - \frac{w_2}{v} \frac{2}{K} \frac{\partial}{\partial z_1} \left(\frac{\partial}{\partial z_1} [U(z_1, z_2) - U(z_1, 0)] v \right) \tag{9.120}$$

[9] Here

$$w(z_1, z_2, t) = w_1(z_1, z_2)\, e^{-\lambda t},$$

as in (9.74).

Near the boundary, the expression on the right takes small values. In fact, according to (9.117),

$$w_2(z_1,0) = 0 \quad \text{for} \quad z_1 < 0, \tag{9.121}$$

and hence

$$w_2(z_1,z_2) = \beta(z_1)z_2 + O(z_2^2) \quad (z_1 < 0). \tag{9.122}$$

where $\beta(z_1)$ is some function. Substituting (9.122) into the right-hand side of (9.120), we obtain

$$\frac{\partial^2 w_2}{\partial z_2^2} + \frac{2}{K}\frac{\partial}{\partial z_2}\left(\frac{\partial U}{\partial z_2}w_2\right) = -B(z_1)z_2 + O(z_2^2), \tag{9.123}$$

where

$$B(z_1) = \frac{d^2\beta(z_1)}{dz_1^2} - \frac{2}{K}\frac{\partial U(z_1,0)}{\partial z_1}\frac{d\beta(z_1)}{dz_1}, \tag{9.124}$$

because of (9.120), (9.119) and (9.122). Integration of (9.123) gives

$$\frac{\partial w_2}{\partial z_2} + \frac{2}{K}\frac{\partial U}{\partial z_2}w_2 = \beta(z_1) - B(z_1)\frac{z_2^2}{2} + \ldots, \tag{9.125}$$

where the constant of integration has been chosen to be $\beta(z_1)$, in keeping with (9.122). Integrating the first-order differential equation (9.125), with the boundary condition (9.121), we find that

$$w_2(z_1,z_2) = e^{-2U/K}\int_0^{z_2} e^{2U/K}\left[\beta(z_1) - B(z_1)\frac{z_2^2}{2} + \ldots\right]dz_2. \tag{9.126}$$

The integral appearing in (9.126) can be evaluated by using the relation

$$\begin{aligned} U(z_1,z_2) &= U(z_1,0) - \frac{\varepsilon\omega_0}{4}\left(1 - \frac{2z_1^2 + z_2^2}{2A_0^2}\right)z_2^2 \\ &= U(z_1,0) - \frac{\varepsilon\omega_0}{4}\left(1 - \frac{z_1^2}{A_0^2}\right)z_2^2 + O(z_2^4), \end{aligned}$$

implied by the form of the potential function (9.105). Therefore, introducing the notation

$$\alpha^2(z_1) = \frac{\varepsilon\omega_0}{K}\left(1 - \frac{z_1^2}{A_0^2}\right) \quad (z_1^2 < A_0^2). \tag{9.127}$$

we have

$$w_2(z_1,z_2)=\exp\left\{-\frac{2}{K}U(z_1,z_2)+\frac{2}{K}U(z_1,0)\right\}$$
$$\times\int_0^{z_2}e^{-\alpha^2z_2^2/2}\left(\beta-\frac{Bz_2^2}{2}+\dots\right)dz_2$$
$$\approx\exp\left\{-\frac{2}{K}[U(z_1,z_2)-U(z_1,0)]\right\}\sqrt{\frac{\pi}{2}}\left(\frac{\beta}{\alpha}-\frac{B}{2\alpha^3}+\dots\right)$$
$$\text{if}\quad \alpha z_2\gg 1. \qquad (9.128)$$

It follows from (9.118), (9.119) and (9.128) that the density w_1 is given by the formula

$$w_1(z_1,z_2)=\sqrt{\frac{\pi}{2}}\left[\frac{\beta(z_1)}{\alpha(z_1)}-\frac{B(z_1)}{2\alpha^3(z_1)}+\dots\right]e^{-2U/K}$$
$$\approx\sqrt{\frac{\pi}{2}}\,\frac{\beta(z_1)}{\alpha(z_1)}e^{-2U/K}, \qquad (9.129)$$

in terms of the function $\beta(z_1)$ describing the gradient of the probability distribution near the boundary. The last approximation anticipates the fact that we are only interested in the region where

$$\frac{B}{\alpha^3}\ll\frac{\beta}{\alpha} \qquad (9.130)$$

[see the discussion below formula (9.134)].

Thus all that remains is to determine the function $\beta(z_1)$. This is the key quantity, since the desired frequency (9.76) can be expressed in terms of $\beta(z_1)$ by using (9.116), (9.118) and (9.122):

$$\Lambda=\frac{K}{2}\int_{-\infty}^{0}v(z_1)\beta(z_1)dz_1. \qquad (9.131)$$

To find $\beta(z_1)$, we use the fact (already mentioned) that the density (9.118) is close to the stationary distribution (9.108) in the region where most of the probability accumulates. Comparing (9.108) and

(9.129) [with only the first term retained], we see at once that

$$\beta(z_1) = \sqrt{\frac{2}{\pi}}\,\frac{\alpha(z_1)}{N}. \tag{9.132}$$

Therefore, using (9.131) and (9.127), we finally obtain the following formula for the rate at which the representative point intersects the half-line $z_1 < 0$:

$$\Lambda = \frac{1}{N}\int_{z_1<0} \exp\left\{-\frac{2}{K}\,U(z_1,0)\right\}\sqrt{1-\frac{z_1^2}{A_0^2}}\,dz_1. \tag{9.133}$$

Because of the exponential factor in (9.133), the contribution to the integral comes mainly from the narrow region [near the point $P_2 = (-A_0a_2,0)$] where[10]

$$\frac{2}{K}\,U(z_1,0) - \frac{2}{K}\,U(-A_0a_2,0) \sim 1. \tag{9.134}$$

The behavior of the potential function (9.115) near P_2 can be approximated by

$$U(z_1,z_2) \approx U_2 + \tfrac{1}{2}\delta_2(z_1 + A_0a_2)^2 - \tfrac{1}{2}\delta_2' z_2^2. \tag{9.135}$$

where the quantities

$$\delta_2 = \frac{\varepsilon\omega_0}{2}(3a_2^2 - 1), \quad \delta_2' = \frac{\varepsilon\omega_0}{2}(1 - a_2^2) > 0 \tag{9.136}$$

play the same role for the point P_2 as the quantities δ and δ' play for the point P_1 [cf. (9.111)]. Using (9.135), we write (9.134) as

$$\frac{\delta_2}{K}(z_1 + A_0a_2)^2 \sim 1. \tag{9.137}$$

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[10] Because of (9.135), this justifies the approximation

$$\Lambda \approx \frac{1}{N}e^{-2U_2/K}\int_{z_1<0} e^{-\delta_2(z_1+A_0a_2)^2/K}\sqrt{1-\frac{z_1^2}{A_0^2}}\,dz_1.$$

It follows from (9.135)–(9.137) that if $z_1 \sim A_0$, then

$$B \sim \frac{\beta}{A_0^2} + \frac{2}{K}\frac{\partial U(z_1,0)}{\partial z_1}\frac{\beta}{A_0} \approx \frac{\beta}{A_0^2} + \frac{2\delta_2}{K}(z_1 + A_0 a_2)\frac{\beta}{A_0}$$
$$\sim \frac{\beta}{A_0^2} + \frac{\beta}{A_0}\sqrt{\frac{\delta_2}{K}} \sim \frac{\beta}{A_0^2} + \frac{\beta}{A_0}\sqrt{\frac{\varepsilon\omega_0}{K}},$$

and hence, as anticipated by (9.130),

$$\frac{B}{\alpha^3} \sim \frac{\beta}{\alpha}\left(\frac{K}{\varepsilon\omega_0 A_0^2} + \sqrt{\frac{K}{\varepsilon\omega_0 A_0^2}}\right) \ll \frac{\beta}{\alpha},$$

where we have used (9.127) and (9.110). It is not hard to see that the whole calculation depends on the assumption that $\Delta_s/\varepsilon\omega_0$ is not too small, since otherwise the variation of the factor

$$\sqrt{1 - \frac{z_1^2}{A_0^2}}$$

in (9.133) cannot be neglected.

Substituting (9.135) into (9.133) and integrating the result between infinite limits [this is possible because of the condition (9.110)], we obtain

$$\Lambda = \frac{1}{N}\sqrt{\frac{\varepsilon\omega_0 K}{2\pi}}\sqrt{\frac{\pi K}{\delta_2}}\sqrt{1 - a_2^2}\, e^{-2U_2/K} = \frac{K}{N}\sqrt{\frac{\delta_2'}{\delta_2}}e^{-2U_2/K}. \qquad (9.138)$$

Using the expression (9.112) for N, we find that

$$\Lambda = \frac{1}{\pi}\sqrt{\frac{\delta\delta'\delta_2'}{\delta_2}}e^{-2(U_2-U_1)/K}, \qquad (9.139)$$

where

$$\frac{\delta\delta'\delta_2'}{\delta_2} = \left(\frac{\varepsilon\omega_0}{2}\right)^2 \frac{3a_1^2 - 1}{3a_2^2 - 1}(a_1^2 - 1)(1 - a_2^2).$$

As the amplitude of the external signal and hence $E/2\varepsilon A_0$ is increased, i.e., as $3p$ is decreased, the point P_2 (see Figure 32) rises higher and higher, and eventually disappears. If the external amplitude is not too large, so that the saddle point P_2 lies below the origin O ($U_2 < 0$), the vast majority of interruptions of synchronization take place along

the bottom of the gutter through the point P_2, as described by formula (9.138). However, for the value

$$3p = 3.8, \quad \text{i.e.,} \quad \frac{E}{2\varepsilon A_0} = \frac{\Delta_s}{\varepsilon\omega_0} = 0.135.$$

the point P_2 has the same height as the origin ($U_2 = 0$), and then we must allow for the possibility of the representative point intersecting the half-line $z_1 < 0$ ($z_2 = 0$) near the origin. Whereas previously the antisynchronous phase $\varphi = \pi$ was achieved most often with amplitude $A_0 a_2$, it is now achieved just as often with very small amplitudes. For $3p < 3.8$, the vast majority of interruptions of synchronization occur with very small amplitudes and have the following appearance: The amplitude of the oscillations decreases in a fluctuational fashion until it almost vanishes, and then after a certain amount of time it builds up again to its previous values. A study of the oscillations during the interval of small amplitudes reveals that in some cases there is a missing period or an extra period. But this anomaly is not very noticeable, because of the smallness of the amplitudes at the moment of interruption of synchronization, and hence such interruptions of synchronization are less important than those occurring with amplitude $A_0 a_2$.

Two kinds of interruptions of synchronization are possible for $3 < p < 3.8$, both through the amplitude $A_0 a_2$ and through small amplitudes, with the latter predominating. As the synchronizing amplitude is made even larger ($3p < 3$), there is no longer any "preferred amplitude" at the moment when the oscillations have the antisynchronous phase, since (9.113) now has only one real root, and in fact, the smaller the amplitude, the larger its probability. The "last" preferred amplitude of the antisynchronous phase occurs for $3p \approx 3$ and equals $A_2 = A_0/\sqrt{3} = A_1/2$, but is effectively masked by the mass of interruptions occurring with other amplitudes.

We again consider the case where $3p$ is not too small and the antisynchronous phase occurs as a rule with amplitude $A_2 = A_0 a_2$ ($A_0 > A_2 > A_0/\sqrt{3}$, $A_1 > A_2 > A_1/2$). Once the phase $\varphi = \pi$ is achieved, in half the cases the phase subsequently "falls back" to the synchronous value, while in the other half the phase "goes over the hump" (the

saddle point), thereby traversing a loop around the origin and acquiring an increment of 2π as it returns to the synchronous value. Thus the frequency of phase jumps equal to 2π is given by $\Lambda/2$. Similarly, the frequency of phase jumps equal to -2π is also given by $\Lambda/2$. Therefore the total frequency of phase jumps of either sign equals Λ. Introducing a diffusion coefficient for the number of periods as in Sec. 3, we obtain the formula

$$D_0 = \Lambda = \frac{1}{\pi}\sqrt{\frac{\delta\delta'\delta_2'}{\delta_2}}\, e^{-2(U_2-U_1)/K}, \tag{9.140}$$

generalizing formula (9.99) [with $\Delta=0$] to the case where the synchronizing signal has large amplitude.

The number of interruptions of synchronization increases rapidly as the mismatch is made different from zero. We conclude this section by giving a brief approximate treatment of the case of large mismatch $\Delta \sim \Delta_s$. In this case, the points P_1 and P_2 no longer lie in the yz_1-plane, and in fact their positions can be determined by setting $\dot{z}_1=\dot{z}_2=0$ and $\xi_1=\xi_2=0$ in (9.103). In Sec. 1 we derived formulas (9.11) and (9.12) determining the polar coordinates $r=A_0a_1$, $\varphi=\varphi_1$ of (the projection of) the point P_1. The coordinates a_2, φ_2 of the second point P_2 can be found from the same equations, but P_2 does not exist if (9.12) has only one root.

On p. 228 we considered the radial and azimuthal "stiffnesses" at the point P_1, i.e.,

$$\delta = \frac{\varepsilon\omega_0}{2}(3a_1^2-1), \quad \delta' = \frac{\varepsilon\omega_0}{2}(a_1^2-1)$$

[cf. (9.16)]. The corresponding radial and azimuthal coefficients for the point P_2 (if it exists) are given by the analogous expressions

$$\delta_2 = \frac{\varepsilon\omega_0}{2}(3a_2^2-1), \quad \delta_2' = \frac{\varepsilon\omega_0}{2}(1-a_2^2),$$

which agree with (9.135). To generalize (9.140) to the case of nonzero mismatch, we must first find the new values of a_1, a_2 and then substitute

them into the formulas for δ, δ', δ_2, δ_2'. Moreover, the line integral

$$-\int_{P_1}^{P_2}(\langle \dot{z}_1\rangle\, dz_1 + \langle \dot{z}_2\rangle\, dz_2),$$

previously equal to the potential difference $U_2 - U_1$ [figuring in (9.140)], now becomes

$$U_2 - U_1 + \Delta \int_{P_1}^{P_2}(z_2\, dz_1 - z_1\, dz_2), \tag{9.141}$$

after taking account of the terms $-\Delta z_2$ and Δz_1 in (9.103).

An exact solution of the problem for nonzero mismatch is very difficult, because of the nonpotential character of the "velocity fields" $\langle \dot{z}_1\rangle$ and $\langle \dot{z}_2\rangle$. Here we sidestep these difficulties and give only a very approximate estimate, sufficient for practical purposes. Although the line integral in (9.141) actually depends on the path of integration, we set

$$\begin{aligned}\int_{P_1}^{P_2}(z_2\, dz_1 - z_1\, dz_2) &= -\int_{(A_1,\varphi_1)}^{(A_2,\varphi_2)} A^2\, d\varphi \\ &= -A_{av}^2(\varphi_2 - \varphi_1) \approx -A_0^2 a_1 a_2(\varphi_2 - \varphi_1),\end{aligned} \tag{9.142}$$

where $z_1 = A\cos\varphi_1$, $z_2 = A\cos\varphi_2$ and A_{av} is an average value of the amplitude, estimated as the geometric mean of A_1 and A_2. Similarly, going from P_1 to P_2 in the opposite direction, we find that

$$\int_{P_1}^{P_2}(z_2\, dz_1 - z_1\, dz_2) \approx -A_0^2 a_1 a_2(2\pi - \varphi_2 + \varphi_1). \tag{9.143}$$

Next, using (9.142) and (9.143) to replace $U_2 - U_1$ in (9.141), where $D_0 = 2G_+ = 2G_-$, we obtain

$$\begin{aligned}G_+ &= \frac{1}{2\pi}\sqrt{\frac{\delta'\delta_2'\delta}{\delta_2}}\exp\left\{-\frac{2}{K}(U_2 - U_1) + \frac{2\Delta}{K}A_0^2 a_1 a_2(\varphi_2 - \varphi_1)\right\} \\ &= \frac{1}{2\pi}\sqrt{\frac{\delta'\delta_2'\delta}{\delta_2}}\exp\left\{-\frac{2}{K}(U_2 - U_1) + a_1 a_2 \pi D \right. \\ &\qquad \left. - a_1 a_2(\pi - \varphi_2 + \varphi_1)D\right\}.\end{aligned}$$

A similar calculation of G_-, based on (9.143), gives

$$G_- = \frac{1}{2\pi}\sqrt{\frac{\bar{\delta}'\bar{\delta}_2'\bar{\delta}}{\delta_2}}\exp\left\{-\frac{2}{K}(U_2-U_1)-\frac{2\Delta}{K}A_0^2a_1a_2(2\pi-\varphi_2+\varphi_1)\right\}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\delta'\delta_2'\delta}{\delta_2}}\exp\left\{-\frac{2}{K}(U_2-U_1)-a_1a_2\pi D - a_1a_2(\pi-\varphi_2+\varphi_1)D\right\}.$$

But as we already know,

$$D_0 = G_+ + G_-, \quad \langle\omega\rangle - \omega_s = 2\pi(G_+ - G_-),$$

which leads to the following formulas corrected for the effects of nonzero mismatch:

$$D_0 = \frac{1}{\pi}\sqrt{\frac{\delta'\bar{\delta}_2'\delta}{\delta_2}}\cosh(a_1a_2\pi D) \times\exp\left\{-\frac{2}{K}(U_2-U_1)-a_1a_2(\pi-\varphi_2+\varphi_1)D\right\},$$

$$\langle\omega\rangle-\omega_s = 2\sqrt{\frac{\delta'\bar{\delta}_2'\delta}{\delta_2}}\sinh(a_1a_2\pi D) \times\exp\left\{-\frac{2}{K}(U_2-U_1)-a_1a_2(\pi-\varphi_2+\varphi_1)D\right\}. \tag{9.144}$$

In particular, we now have

$$\frac{\langle\omega\rangle-\omega_s}{2\pi D_0} = \tanh(a_1a_2\pi D),$$

instead of the formula

$$\frac{\langle\omega\rangle-\omega_s}{2\pi D_0} = \tanh\pi D$$

implied by (9.94). The formulas (9.144) generalize formulas (9.93) and (9.99), to which they reduce for $\Delta_s/\varepsilon\omega_0 \ll 1$.

Supplement

★ We now clarify the remarks made at the end of Sec. 4, concerning the spectral structure of the signal generated by a synchronized oscillator. Suppose the phase jumps in the forward direction occur at times t_i^+ and have a Poisson distribution with density G_+. Let the analogous quantities for the backward jumps be t_i^- and G_-. Then, neglecting small phase fluctuations, we have

$$\dot{\varphi}(t) = \sum_i F_+(t - t_i^+) - \sum_i F_-(t - t_i^-), \tag{9.145}$$

where F_+ and F_- are certain "pulses" (see below) such that

$$\int F_\pm(\theta)\, d\theta = 2\pi. \tag{9.146}$$

Let

$$\xi_\pm(t) = \sum_i \delta(t - t_i^\pm) \tag{9.147}$$

be the corresponding "random density functions" [cf. (I.6.21)], described by the generating functionals

$$L_\pm[v(t)] = \exp\left\{\int_{-\infty}^{\infty} G_\pm v(t)\, dt\right\} \tag{9.148}$$

[cf. (I.6.31)]. Then (9.145) implies

$$\varphi(\tau) - \varphi(0) = \int_0^\tau dt \int_{-\infty}^{\infty} [F_+(t - s)\xi_+(s) - F_-(t - s)\xi_-(s)]\, ds. \tag{9.149}$$

By analogy with (I.7.16) and (8.32), the correlation function of the generated signal can be written as

$$\langle x\overline{x}_\tau\rangle = \tfrac{1}{2}A_0^2 \operatorname{Re}\langle e^{-i\varphi(\tau) + i\varphi(0) - i\omega_0\tau}\rangle. \tag{9.150}$$

The next step is to substitute (9.149) into (9.150). Together, (I.6.24) and (9.148) give

$$\left\langle \exp\left\{i\int u(s)\xi(s)\, ds\right\}\right\rangle = \exp\left\{G_\pm \int [e^{iu\tau} - 1]\, ds\right\}. \tag{9.151}$$

It then follows from (9.149) and (9.150) that

$$\langle xx_\tau\rangle = \tfrac{1}{2}A_0^2\,\mathrm{Re}\left[\exp\left\{G_+\int_{-\infty}^{\infty} e^{-i\int_0^s F_+(t-s)\,dt} - 1\right]ds + G_-\int_{-\infty}^{\infty} e^{i\int_0^s F_-(t-s)\,dt} - 1\right]ds - i\omega_s t\Big\}\Big], \quad \tau \geqslant 0. \tag{9.152}$$

Here we regard the functions $\xi_+(t)$ and $\xi_-(t)$ as statistically independent, a natural assumption in the Poisson approximation. The spectral density is now found by taking the Fourier transform of (9.152). By analogy with (I.7.13) and (8.88), we have

$$S[x;\omega] = \tfrac{1}{4}A_0^2 S_1(\omega - \omega_s), \tag{9.153}$$

where

$$S_1(\nu) = 2\int_{-\infty}^{\infty} f_+(\tau)f_-(\tau)e^{i\nu\tau}\,d\tau = 4\,\mathrm{Re}\left\{\int_0^{\infty} f_+(\tau)f_-(\tau)e^{i\nu\tau}\,d\tau\right\} \tag{9.154}$$

[cf. (9.157)] and

$$f_\pm(\tau) = \exp\left\{G_\pm\int_{-\infty}^{\infty}\left[e^{\mp i\int_0^s F(t-s)dt} - 1\right]ds, \quad \tau \geqslant 0. \right. \tag{9.155}$$

The functions $F_\pm$ can be determined from equations of the form

$$F_+(\theta) = \Delta - \Delta_s \sin\int_c^\theta F(\theta')\,d\theta',$$

obtained by averaging (9.43), but qualitative estimates can be made by using the approximation

$$F_+(\theta) = F_-(\theta) = \begin{cases} \dfrac{2\pi}{\tau_0} & \text{for} \quad 0 \leqslant \theta \leqslant \tau_0, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\tau_0 \sim \frac{1}{\Delta_s}\ln D_s$$

[cf. (9.95)]. Then

$$\int_0^\tau F_\pm(t-s)\,dt = \frac{2\pi}{\tau_0}\begin{cases} s+\tau_0 & \text{for} \quad -\tau_0 \leqslant s \leqslant -\tau_0+\tau, \\ \tau & \text{for} \quad -\tau_0+\tau \leqslant s \leqslant 0, \\ \tau - s & \text{for} \quad 0 \leqslant s \leqslant \tau, \\ 0 & \text{otherwise} \end{cases}$$

if $\tau \leqslant \tau_0$, while

$$\int_0^\tau F_\pm(t-s)\,dt = \frac{2\pi}{\tau_0}\begin{cases} s+\tau_0 & \text{for} \quad -\tau_0 \leqslant s \leqslant 0, \\ \tau_0 & \text{for} \quad 0 \leqslant s \leqslant \tau-\tau_0, \\ \tau - s & \text{for} \quad \tau-\tau_0 \leqslant s \leqslant \tau, \\ 0 & \text{otherwise} \end{cases}$$

if $\tau \geqslant \tau_0$. Substituting these functions into (9.155) and integrating with respect to s, we obtain

$$f_\pm(\tau) = \exp\left\{2G_\pm \int_0^\tau \left(e^{\mp 2\pi i s/\tau_0} - 1\right)ds + G_\pm(\tau_0-\tau)\left(e^{\mp 2\pi i\tau/\tau_0} - 1\right)\right\}$$

$$\text{for} \quad 0 \leqslant \tau \leqslant \tau_0,$$

$$f_\pm(\tau) = \exp\left\{2G_\pm \int_0^{\tau_0} \left(e^{\mp 2\pi i s/\tau_0} - 1\right) ds\right\} \quad \text{for} \quad \tau \geqslant \tau_0,$$

and hence

$$f_+(\tau)f_-(\tau)$$

$$= \exp\left\{(G_+ + G_-)\left[\frac{\tau_0}{\pi}\sin\frac{2\pi\tau}{\tau_0} + (\tau_0-\tau)\cos\frac{2\pi\tau}{\tau_0} - \tau - \tau_0\right]\right.$$

$$\left. - i(G_+ - G_-)\left[\frac{\tau_0}{\pi}\left(1-\cos\frac{2\pi\tau}{\tau_0}\right) + (\tau_0-\tau)\sin\frac{2\pi\tau}{\tau_0}\right]\right\}$$

$$\text{for} \quad 0 \leqslant \tau \leqslant \tau_0,$$

$$f_+(\tau)f_-(\tau) = \exp\{-2(G_+ + G_-)\tau_0\} \quad \text{for} \quad \tau \geqslant \tau_0. \tag{9.156}$$

The function $f_+(\tau)f_-(\tau)$ is continued into the region $\tau < 0$ by using the formula

$$f_+(-\tau)f_-(-\tau) = f_+^*(\tau)f_-^*(\tau), \tag{9.157}$$

implied by

$$\langle e^{-i\varphi(-\tau)+i\varphi(0)}\rangle = \langle e^{i\varphi(\tau)-i\varphi(0)}\rangle.$$

The presence in $f_+(\tau)f_-(\tau)$ of imaginary terms (and hence of terms odd in τ) testifies to the asymmetry of the spectral density $S_1(\nu)$ about $\nu=0$, i.e., of the spectral density $S[x;\omega]$ about $\omega=\omega_s$, stemming from the fact that G_+-G_- is nonzero. It is clear from (9.154) and (9.156) that $S[x;\omega]$ has a delta-function peak

$$\pi A_0^2\, e^{-2(G_+ + G_-)\tau_0}\delta(\omega-\omega_s),$$

which contributes an amount of power

$$\langle x^2\rangle_1 = \tfrac{1}{2}A_0^2\, e^{-2(G_+ + G_-)\tau_0}.$$

The remaining power

$$\begin{aligned}\langle x^2\rangle_2 &= \langle x^2\rangle - \langle x^2\rangle_1 = \\ &= \tfrac{1}{2}A_0^2\left[1 - e^{-2(G_+ + G_-)\tau_0}\right] \approx A_0^2(G_+ + G_-)\tau_0 \sim A_0^2\, e^{-\delta u}\end{aligned}$$

is the total power of the continuous component of the spectral density, whose bandwidth, according to (9.156), is of order

$$\frac{1}{\tau_0} \sim \Delta_s.$$

Let ω_c be the "center frequency" of the continuous component, i.e., its center of gravity. Then the mean frequency $\langle\omega\rangle$, equal to the center of gravity of the *total* spectral density, is a weighted average of the discrete and continuous components:

$$\begin{aligned}\langle\omega\rangle &= \left[1 - 2(G_+ + G_-)\tau_0\right]\omega_s + 2(G_+ + G_-)\tau_0\omega_c \\ &= \omega_s + 2(G_+ + G_-)\tau_0(\omega_c - \omega_s).\end{aligned}$$

Using (9.93), we find that the continuous component is shifted from the discrete component by an amount

$$\omega_c - \omega_s = \frac{\langle\omega\rangle - \omega_s}{2\tau_0(G_+ + G_-)} = \frac{\pi}{\tau_0}\frac{G_+ - G_-}{G_+ + G_-} \sim \Delta_s \tanh \pi D$$

as asserted on p. 258. ★

CHAPTER 10

Parametric Oscillations

The adjective "parametric" is applied to phenomena and processes which are described by differential equations with variable parameters, more exactly, whose differential equations contain linear terms with time-dependent coefficients. The equations as a whole can be either linear or nonlinear, and oscillations can be excited in the system by both regular and irregular oscillations of the parameters. It is more convenient to study the conditions for excitation of such oscillations by using the example of a linear system, but nonlinearity must be taken into account in order to calculate the amplitude of the oscillations. However, even the linear theory of parametric excitation is intimately connected with the theory of nonlinear oscillations, both in subject matter and in methodology. It is for this reason that we follow tradition and study parametric oscillations under the heading of nonlinear oscillations.

As a very simple example of an electrical system with variable parameters, consider the oscillatory circuit shown in Figure 33, where

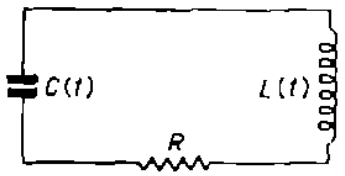

Fig. 33. An oscillatory circuit with variable parameters

the capacitance C and the inductance L vary in time. The oscillations in this system are described by the equation

$$\ddot{y} + 2\delta(t)\dot{y} + k(t)y = 0, \tag{10.1}$$

where y is the current or voltage in any element, and $k(t) = 1/LC$ and $2\delta(t) = R/L$ are parameters varying in time, because of instability of the inductance and capacitance. This instability can be produced either by spontaneous internal fluctuations or by some externally applied variable voltage.

Various special effects may be used to produce controlled variation of the parameters, e.g., the dependence of interelectrode capacitance on voltages applied to other grids, the dependence of the magnetic permeability of ferromagnetic materials on the magnetic field intensity, and so on. Parametric systems with controlled frequency changes resemble frequency-modulation systems, from which they differ only by the rate of change of the frequency. An important class of parametric systems consists of mechanical systems, e.g., a pendulum executing horizontal oscillations while the point of suspension undergoes vertical displacements, transverse oscillations of a hinged beam or of a fastened string subject to time-dependent forces, etc.

As already noted, the character of the parameter changes may be either regular or irregular. Mixed cases are also possible, where fluctuational excitations are imposed on regular (say, harmonic) oscillations. Here we restrict ourselves to the case where the intensity of the parameter changes is small, so that the oscillations are close to being harmonic. As applied to the last term in equation (10.1), this means that $k(t)$ is the sum of a large constant term and a small variable term, i.e.,

$$k(t) = k_0^2 + \varepsilon k_1(t), \tag{10.2}$$

where ε is a small parameter whose presence allows us to use the asymptotic methods considered earlier and to achieve accuracies of various orders in ε. Using the first approximation (of the first order in ε), we can investigate the fundamental parametric resonance, in which the system responds to components of frequency $2\omega_0$ in the spectrum of the function $k(t)$, but investigation of higher-order

resonances requires the use of higher approximations. The present treatment is confined to a study of the fundamental parametric resonance and of the equations obtained in the first approximation (which coincide with the truncated equations of Chap. 4, Sec. 3).

1. Linear Parametric Oscillations. The Basic Equations.

Making the change of variable

$$x(t) = \exp\left\{\int_0^t \delta(t')\,dt'\right\} y(t), \tag{10.3}$$

we transform the linear equation (10.1) into the form

$$\ddot{x} + [k(t) - \delta^2(t) - \dot{\delta}(t)]\,x = 0. \tag{10.4}$$

From $k(t)-\delta^2(t)-\dot{\delta}(t)$ we subtract out the constant component, denoting it by ω_0^2, and then denote the remaining expression by $\omega_0^2\xi(t)$:

$$k(t) - \delta^2(t) - \dot{\delta}(t) = \omega_0^2[1 + \xi(t)]. \tag{10.5}$$

If $k(t)-\delta^2(t)$ is a stationary random process, we choose

$$\omega_0^2 = \langle k(t) - \delta^2(t)\rangle,$$

so that ξ has zero mean value $\langle\xi\rangle = 0$. With the notation (10.5), equation (10.4) becomes

$$\ddot{x} + \omega_0^2 x = -\omega_0^2 x\xi(t). \tag{10.6}$$

Using the same method as in Chap. 4, Sec. 3, we define the amplitude and the phase of the oscillations by the formulas

$$x = A\cos(\omega_0 t + \varphi), \quad \dot{x} = -\omega_0 A \sin(\omega_0 t + \varphi). \tag{10.7}$$

As before, the amplitude and phase satisfy the equations (4.31), which in the present case, because of (10.6), take the form

$$\dot{A} = -\frac{x\dot{x}}{A}\xi(t), \quad \dot{\varphi} = \omega_0 \frac{x^2}{A^2}\xi(t) \tag{10.8}$$

or

$$\begin{aligned} \frac{\dot{A}}{A} &= \omega_0 \xi(t) \sin(\omega_0 t + \varphi) \cos(\omega_0 t + \varphi), \\ \dot{\varphi} &= \omega_0 \xi(t) \cos^2(\omega_0 t + \varphi), \end{aligned} \tag{10.9}$$

after substituting from (10.7).

The amplitude A figuring in (10.9) is the amplitude of the "fictitious" process $x(t)$, differing by the factor

$$\exp\left\{\int_0^t \delta(t')\,dt'\right\}$$

from the original process $y(t)$ which has the usual direct physical meaning. The "physical" amplitude of the oscillations $y(t)$ equals

$$\exp\left\{-\int_0^t \delta(t')\,dt'\right\} A(t). \tag{10.10}$$

After introducing the special notation

$$u = \ln A - \int_0^t \delta(t')\,dt' \tag{10.11}$$

for the logarithm of (10.10), we can write (10.9) in the form

$$\begin{aligned} \dot{u} &= -\delta(t) + \frac{\omega_0}{2}\xi(t) \sin(2\omega_0 t + 2\varphi), \\ \dot{\varphi} &= \frac{\omega_0}{2}\xi(t) + \frac{\omega_0}{2}\xi(t) \cos(2\omega_0 t + 2\varphi). \end{aligned} \tag{10.12}$$

The equations (10.12) are perfectly exact and valid for arbitrary functions $\xi(t)$ and $\delta(t)$, both random and nonrandom. Of course, we are interested in processes where $\xi(t)$ and $\delta(t)$ are random, or at least contain random components. We shall assume that the smallness conditions

$$\xi(t) \ll 1, \quad \delta(t) \ll \omega_0 \tag{10.13}$$

are satisfied, since otherwise it is difficult to solve the equations (10.12)

(although they remain meaningful) and the very concept of the amplitude and phase of the processes $x(t)$ and $y(t)$ loses its meaning.

We now ask whether or not oscillations are excited in the system, i.e., whether or not the system is stable. It is easy to see that the answer to this question depends on the size of the mean value $\langle u \rangle$, which can be determined from the first of the equations (10.12). Averaging this equation, we find that the inequality

$$\langle \delta(t) \rangle < \frac{\omega_0}{2} \langle \xi(t) \sin (2\omega_0 t + 2\varphi) \rangle \tag{10.14}$$

represents the condition for excitation of parametric oscillations. In fact, if this inequality is not satisfied, then any oscillations present in the system are completely damped out in the course of time, and hence the system is stable. Clearly, the quantity

$$\delta_{cr} = \frac{\omega_0}{2} \langle \xi(t) \sin (2\omega_0 t + 2\varphi) \rangle \tag{10.15}$$

represents a "critical damping coefficient," since the system is excited if $\langle \delta(t) \rangle < \delta_{cr}$ but not if $\langle \delta(t) \rangle > \delta_{cr}$, i.e., the system can no longer be excited after $\langle \delta(t) \rangle$ reaches the value δ_{cr}.

Thus investigation of the stability of a parametric system reduces to calculating the quantity (10.15). This requires the use of the second of the equations (10.12), as well as the first, since the phase $\varphi(t)$ is intimately related to $\xi(t)$. Fortunately, the equation for the phase does not contain $u(t)$, and hence can be considered independently of the first equation.

The trigonometric function $\cos (2\omega_0 t + 2\varphi)$ in the phase equation, which explicitly involves the time, singles out the harmonic component of frequency near $2\omega_0$ from the function $\xi(t)$, a feature which is characteristic of the fundamental parametric resonance. To study this effect in its pure form, suppose the spectral density of the process $\xi(t)$ is appreciably different from zero only in a narrow band

$$|\omega - 2\omega_0| \leq \tfrac{1}{2}\Delta\omega$$

near the frequency $2\omega_0 \gg \Delta\omega$. Then $\xi(t)$ can be written in the form

$$\xi(t) = h(t) \sin [2\omega_0 t + \psi(t)], \tag{10.16}$$

where the amplitude $h(t)$ and phase $\psi(t)$ are slowly varying compared to $\sin 2\omega_0 t$, but can vary rapidly compared to $A(t)$ and $\varphi(t)$. Substituting (10.16) into (10.12), using the formula

$$2 \sin (2\omega_0 t + \varphi) \cos (2\omega_0 t + 2\varphi) = \sin (4\omega_0 t + \psi + 2\varphi) + \sin (\psi - 2\varphi),$$

and dropping oscillatory terms involving $\sin (4\omega_0 t + \psi + 2\varphi)$ and $\sin (2\omega_0 t + \psi)$ [which are unimportant in the first approximation], we obtain the following first-order "simplified" equation:

$$\dot{\varphi} = -\frac{\omega_0}{4} h(t) \sin [2\varphi(t) - \psi(t)]. \tag{10.17}$$

In deriving this equation, which no longer involves the time explicitly, we commit the error of considering only the fundamental parametric resonance. We can transform the first of the equations (10.12), or the expression (10.15) for the critical damping, in the same way, obtaining, for example,

$$\delta_{cr} = \frac{\omega_0}{4} \langle h(t) \cos [2\varphi(t) - \psi(t)] \rangle. \tag{10.18}$$

Unfortunately, even the simple equation (10.17) has no exact solution in quadratures. However, special results can be obtained by using conditions involving the rate of change of the functions $h(t)$ and $\psi(t)$.

2. Narrow-Band Quasi-Harmonic Parametric Oscillations.

As can be seen from (10.17), the phase changes at a rate characterized by the time constant

$$T_{rel} = \frac{1}{\omega_0 \langle h \rangle}, \tag{10.19}$$

which may be called the *phase relaxation time*. In analyzing (10.17), it is important to know whether the functions $h(t)$ and $\psi(t)$ manage to change appreciably during the time T_{rel}. Consider the case where $h(t)$ and $\psi(t)$ are slowly varying, i.e., where the time constants $h/\dot{h}$, $\dot{\psi}/\ddot{\psi}$ and T_ψ are all much greater than the phase relaxation time (T_ψ denotes

the time for a change in ψ of order 1 to accumulate). Letting χ denote the deviation

$$\chi(t) = 2\varphi(t) - \psi(t),$$

we write (10.17) in the form

$$\dot{\chi} + \frac{\omega_0}{2} h(t) \sin \chi = -\dot{\psi}(t). \tag{10.20}$$

If the frequency of the external excitation is slowly varying ($\dot{\psi}/\ddot{\psi} \ll \omega_0 h$), the phase deviation χ will differ only slightly from the "stable instantaneous" value $\chi_0(t)$ determined from the equation

$$\frac{\omega_0}{2} h(t) \sin \chi_0 = -\dot{\psi}, \tag{10.21}$$

obtained from (10.20) by setting $\dot{\varphi} = 0$. Substituting the value

$$\chi_0 = -\arcsin \frac{2\dot{\psi}}{\omega_0 h} \tag{10.22}$$

[implied by (10.21)] into (10.18) instead of $2\varphi - \psi$, we obtain the following expression for the critical damping:

$$\delta_{cr} = \tfrac{1}{4} \langle \sqrt{\omega_0^2 h^2(t) - 4\dot{\psi}^2} \rangle. \tag{10.23}$$

For constant h and $\dot{\psi}$, formula (10.23) coincides with the result of the ordinary nonfluctuational theory of parametric excitation. Thus excitation is possible only if $|2\dot{\psi}| < \omega_0 h$, a condition which we shall assume to be satisfied, since otherwise the quasi-static approach of this section would be inapplicable.

A better approximation to δ_{cr} can be obtained by taking account of the deviation

$$\chi_1 = \chi - \chi_0 \tag{10.24}$$

from the value (10.22). For small deviations, equation (10.20) can be linearized, which gives the following linear equation:

$$\dot{\chi}_1 + \frac{\omega_0}{2} h(t) \cos \chi_0(t) \cdot \chi_1 = -\dot{\chi}_0. \tag{10.25}$$

Assuming that the process began in the remote past, we can solve

(10.25) by the usual method, obtaining

$$\chi_1 = -\int_{-\infty}^{t} e^{-\int_{t'}^{t}\gamma\,dt}\,\dot{\chi}_0(t')\,dt',$$
$$\gamma = \frac{\omega_0}{2}\,h(t)\cos\chi_0 = \sqrt{\tfrac{1}{4}\omega_0^2 h^2 - \dot{\psi}^2}. \tag{10.26}$$

Correspondingly, only the principal terms should be retained in the expression (10.18). Thus, dropping cubic and higher-order terms, we find that

$$\delta_{cr} = \frac{\omega_0}{4}\left\langle h\cos\chi_0\left(1-\frac{\chi_1^2}{2}\right)\right\rangle - \frac{\omega_0}{4}\langle h\sin\chi_0\cdot\chi_1\rangle$$
$$= \frac{1}{2}\left\langle\sqrt{\tfrac{1}{4}\omega_0^2 h^2 - \dot{\psi}^2}\left(1-\frac{\chi_1^2}{2}\right)\right\rangle + \frac{1}{2}\langle\dot{\psi}\chi_1\rangle. \tag{10.27}$$

Substituting (10.26) and (10.22) into (10.27), we can calculate δ_{cr}, given the statistical properties of the original processes $h(t)$ and $\psi(t)$. These considerations are based on the use of the quasi-static method presented in Chap. 5, Sec. 3, and in fact, equation (10.17) clearly resembles the second of the equations (5.51).

If the condition

$$T_\psi \gg \frac{1}{\omega_0 h} \tag{10.28}$$

holds with overwhelming probability, then the quasi-static treatment of parametric excitation can be simplified. In this case, χ is small enough to allow us to replace $\sin\chi$ by χ in equation (10.20). This leads to the linear equation

$$\dot{\chi} + \frac{\omega_0}{2}\,h\chi = -\dot{\psi}, \tag{10.29}$$

with solution

$$\chi(t) = -\int_{-\infty}^{t} e^{-\frac{1}{2}\omega_0\int_{t'}^{t} h\,dt}\,\dot{\psi}(t')\,dt'. \tag{10.30}$$

We can now replace $\cos\chi$ by $1-(\chi^2/2)$ in calculating the damping

(10.18), i.e.,

$$\delta_{cr} = \frac{\omega_0}{4}\left\langle h(t)\left(1 - \frac{\chi^2}{2}\right)\right\rangle. \tag{10.31}$$

where this expression can be evaluated by substituting from (10.30) and carrying out the average.

For example, consider the special case where the amplitude of the parametric oscillations is constant, but the phase undergoes a "completely random walk," so that $\dot{\psi}$ is a delta-correlated random process with zero mean and known intensity κ:

$$\langle \dot{\psi}\dot{\psi}_\tau \rangle = \kappa\,\delta(\tau).$$

Then formula (10.30) becomes

$$\chi(t) = -\int_{-\infty}^{t} e^{-\frac{1}{2}\omega_0 h(t-t')}\dot{\psi}(t')\,dt',$$

and hence

$$\langle \chi^2 \rangle = \frac{\kappa}{\omega_0 h}, \tag{10.32}$$

because of the form of the correlation function of $\dot{\psi}$. It follows from (10.31) that the critical damping is

$$\delta_{cr} = \frac{\omega_0 h}{4}\left(1 - \frac{\kappa}{2\omega_0 h}\right). \tag{10.33}$$

Therefore the parametric system is stable if

$$\delta > \frac{\omega_0 h}{4}\left(1 - \frac{\kappa}{2\omega_0 h}\right), \tag{10.34}$$

and unstable otherwise.

The first term $\omega_0 h/4$ in formulas (10.33) and (10.34) can be obtained from the ordinary nonfluctuational theory. The second term, which is $\kappa/2\omega_0 h$ times smaller than the first, describes the effect of the fluctuational instability of parametric oscillations on the excitation condition. Thus we see that instability of the parametric oscillations inhibits parametric excitation and is equivalent to decreasing the amplitude of the oscillations. Of course, the expression (10.33) is not

exact, and making it more exact would introduce a correction, whose relative size is in turn approximately $\kappa/2\omega_0 h$ times smaller than the second term. Formula (10.33) breaks down for $\kappa/2\omega_0 h \gtrsim 1$, since then the conditions for applicability of the quasi-static approach no longer hold.

The above method is not applicable when the amplitude, phase and frequency change so rapidly that they manage to vary appreciably during a time of the order of the phase relaxation time (10.19), since in this case it is difficult to solve (10.17) or (10.20). However, in the opposite extreme case, where the fluctuations are sufficiently rapid, one can apply the stochastic method based on replacing the actual fluctuations $\xi(t)$ by a delta-correlated random process.

3. Rapid Parametric Fluctuations. Application of the Stochastic Method

We now assume that the phase relaxation time is much larger than the correlation time of the parametric fluctuations:

$$T_{rel} \gg \tau_{cor}. \tag{10.35}$$

Here T_{rel} is given by formula (10.19), and τ_{cor} is related to the rate of change of the functions $\psi(t)$ and $h(t)$, if we represent $\xi(t)$ in the form (10.16). However, there is no longer any particular need to use this representation, and we can consider the equations (10.12) directly, regarding τ_{cor} as the correlation time of the original process $\xi(t)$.

It should be noted that the condition for applicability of the stochastic method is connected with the condition that the parametric fluctuations be small. In fact, if we decrease $\xi(t)$, leaving its rate of change the same, the phase relaxation time increases until it eventually enters the region of applicability of the stochastic method, regardless of the size of the correlation time.

In the stochastic approach, we regard the process $\xi(t)$ figuring in the equations (10.12) as a delta-correlated process. Then the phase fluctuations will be a Markov process, described by an appropriate Fokker-Planck equation, which can be written down by using the rules given in Chap. I.4, Sec. 8. Noting that the second of the equations

(10.12) is a special case of equations (I.4.142), and using formulas (I.4.161) and (I.4.162), we obtain the Fokker-Planck equation

$$\dot{w}(\varphi) = -\frac{\partial}{\partial\varphi}[m_2(\varphi)w(\varphi)] + \frac{1}{2}\frac{\partial^2}{\partial\varphi^2}[K_2(\varphi)w(\varphi)]. \tag{10.36}$$

where

$$m_2(\varphi) = \frac{\omega_0^2}{4}\int_{-\infty}^{0}\langle\xi\xi_\tau\rangle\frac{\partial}{\partial\varphi}[1+\cos(2\omega_0 t+2\varphi)] \times [1+\cos(2\omega_0 t+2\omega_0\tau+2\varphi)]\,d\tau. \tag{10.37}$$

$$K_2(\varphi) = \frac{\omega_0^2}{4}\int_{-\infty}^{\infty}\langle\xi\xi_\tau\rangle[1+\cos(2\omega_0 t+2\varphi)] \times [1+\cos(2\omega_0 t+2\omega_0\tau+2\varphi)]\,d\tau. \tag{10.38}$$

The quantity (10.37) is the mean value of the right-hand side of the second of the equations (10.12), i.e., of the expression

$$\frac{\omega_0}{2}\xi(t)[1+\cos(2\omega_0 t+2\varphi)].$$

Since, by definition $\langle\xi\rangle=0$, the fact that $m_2(\varphi)$ is nonzero is due to correlation between $\xi(t)$ and φ in the product $\xi(t)\cos(2\omega_0 t+2\varphi)$.[1]

Similarly, to study the stability of the parametric system, we need the mean value of the right-hand side of the first of the equations (10.12), and here an important role is played by correlation between $\xi(t)$ and φ in the product $\xi(t)\sin(2\omega_0 t+2\varphi)$. The problem of averaging expressions containing a random excitation and a Markov process was studied in Chap. I.4, Sec. 8, where we derived formula (I.4.167). Using this formula, we find that

$$\langle\xi(t)\sin(2\omega_0 t+2\varphi)\rangle = \frac{\omega_0}{2}\int_{-\infty}^{0}\langle\xi\xi_\tau\rangle\frac{\partial}{\partial\varphi}[\sin(2\omega_0 t+2\varphi)] \times[1+\cos(2\omega_0 t+2\omega_0\tau+2\varphi)]\,d\tau, \tag{10.39}$$

a result which closely resembles (10.37).

[1] The validity of (10.37), and of (10.39) as well, can also be established by using the approximation method described on pp. 110–113.

The expressions (10.37)–(10.39) explicitly involve the time, and besides constant components, they contain oscillatory components which vary rapidly with frequencies $2\omega_0$ and $4\omega_0$. Since φ and $w(\varphi)$ change much more slowly, the effect of these oscillatory components is small. In a first approximation, they can simply be neglected, by averaging over a period, as in Chap. 4, Sec. 4. For example, transforming the expression

$$[1 + \cos(2\omega_0 t + 2\varphi)][1 + \cos(2\omega_0 t + 2\omega_0\tau + 2\varphi)]$$

appearing in (10.38) into the form

$$1 + \cos(2\omega_0 t + 2\varphi) + \cos(2\omega_0 t + 2\omega_0\tau + 2\varphi) + \tfrac{1}{2}\cos 2\omega_0\tau + \tfrac{1}{2}\cos(4\omega_0 t + 2\omega_0\tau + 4\varphi),$$

and dropping oscillatory terms (the second, third and fifth terms), we obtain

$$K_2(\varphi) = \frac{\omega_0^2}{4}\int_{-\infty}^{\infty}\langle\xi\xi_\tau\rangle[1 + \tfrac{1}{2}\cos 2\omega_0\tau]\,d\tau = \frac{\omega_0^2}{4}[\kappa(0) + \tfrac{1}{2}\kappa(2\omega_0)], \tag{10.40}$$

where

$$\kappa(\omega) = \int_{-\infty}^{\infty}\langle\xi\xi_\tau\rangle\cos\omega\tau\,d\tau. \tag{10.41}$$

In just the same way, after dropping oscillatory terms in (10.37) and (10.39), we find that

$$m_2(\varphi) = -\frac{\omega_0^2}{4}\int_{-\infty}^{0}\langle\xi\xi_\tau\rangle\sin 2\omega_0\tau\,d\tau, \tag{10.42}$$

$$\langle\xi(t)\sin(2\omega_0 t + 2\varphi)\rangle = \frac{\omega_0}{2}\int_{-\infty}^{0}\langle\xi\xi_\tau\rangle\cos 2\omega_0\tau = \frac{\omega_0}{4}\kappa(2\omega_0). \tag{10.43}$$

It follows from (10.43) that the critical damping is

$$\delta_{cr} = \frac{\omega_0}{2}\langle\xi(t)\sin(2\omega_0 t + 2\varphi)\rangle = \frac{\omega_0^2}{8}\kappa(2\omega_0). \tag{10.44}$$

If $\delta < \delta_{cr}$, then $\langle \dot{u} \rangle > 0$ in the first of the equations (10.12), and consequently parametric excitation takes place.

Besides equation (10.36), corresponding to the second of the equations (10.12), we can also derive a stochastic equation associated with the first of the equations (10.12) and describing the evolution of the distribution $w(u)$. Taking account of (10.44), we can write this equation in the form

$$\dot{w}(u) = (\langle \delta \rangle - \delta_{cr}) \frac{\partial w}{\partial u} + \frac{K_1}{2} \frac{\partial^2 w}{\partial u^2}, \tag{10.45}$$

where

$$K_1 = \frac{\omega_0^2}{8} \kappa(2\omega_0) + \int_{-\infty}^{\infty} \mathbf{K}[\delta, \delta_\tau] \, d\tau \tag{10.46}$$

[cf. (10.38) and (10.40)]. The Fokker-Planck equations (10.45) and (10.36) are equivalent to the fluctuation equations

$$\begin{aligned} \dot{u} &= \delta_{cr} - \langle \delta \rangle + \xi_1, \\ \dot{\varphi} &= m_2 + \xi_2 \end{aligned} \tag{10.47}$$

containing the delta-correlated random processes $\xi_1(t)$ and $\xi_2(t)$, with zero mean values and intensities (10.46) and (10.40), respectively, i.e.,

$$\langle \xi_1 \xi_{1\tau} \rangle = K_1 \delta(\tau), \quad \langle \xi_2 \xi_{2\tau} \rangle = K_2 \delta(\tau).$$

As an example, we now consider the same case as at the end of the preceding section. Let $\xi(t)$ be an oscillatory process whose amplitude h is fixed and whose phase ψ undergoes a "completely random walk":

$$\begin{aligned} &\xi(t) = h \sin[2\omega_0 t + \psi(t)], \\ &\langle \dot{\psi} \rangle = 0, \quad \langle \psi \psi_\tau \rangle = \kappa \delta(\tau). \end{aligned} \tag{10.48}$$

To apply the stochastic method, we must require that the phase deviations be sufficiently strong and hence that κ be large. Thus, during a time of order $1/\omega_0 h$, the phase must "change completely," i.e., a phase shift much greater than π must accumulate. Because of the form of the correlation function of ψ, this condition becomes

$$\frac{\kappa}{\omega_0 h} \gg 1. \tag{10.49}$$

The correlation function $\langle \xi \xi_\tau \rangle$ is easily calculated from a knowledge of the characteristic function of the phase increment after time τ:

$$\Theta(v) = \left\langle \exp\left\{ iv \int_t^{t+\tau} \psi \, dt \right\} \right\rangle = e^{-\kappa \tau v^2/2} \quad (\tau > 0). \qquad (10.\ 0)$$

If we average over the random initial phase, an oscillatory term depending on t drops out of $\langle \xi \xi_\tau \rangle$, leaving

$$\langle \xi \xi_\tau \rangle = \frac{h^2}{2} \langle \cos(2\omega_0 \tau + \psi_\tau - \psi) \rangle . \qquad (10.51)$$

But $\cos(2\omega_0\tau + \psi_\tau - \psi)$ is the real part of $\exp\{i(2\omega_0\tau + \psi_\tau - \psi)\}$, and hence (10.51) is easily calculated by multiplying (10.50) by $e^{2i\omega_0\tau}$, taking the real part and setting $v = 1$. This gives the correlation function

$$\langle \xi \xi_\tau \rangle = \frac{h^2}{2} e^{-\kappa|\tau|/2} \cos 2\omega_0 \tau ,$$

with corresponding spectral density

$$\kappa(\omega) = \frac{h^2 \kappa}{4} \left[\frac{1}{(\frac{1}{2}\kappa)^2 + (\omega - 2\omega_0)^2} + \frac{1}{(\frac{1}{2}\kappa)^2 + (\omega + 2\omega_0)^2} \right]. \qquad (10.52)$$

Substituting (10.52) into (10.44), we find the critical damping

$$\delta_{cr} = \frac{\omega_0^2 h^2}{8} \left(1 + \frac{\kappa^2}{\kappa^2 + 64\omega_0^2} \right). \qquad (10.53)$$

It is interesting to compare this result with the critical damping coefficient (10.33) for the case of weak phase diffusion. According to (10.33) and (10.53), we have the following dependence of δ_{cr} on $x = \kappa/\omega_0 h$ (restricting ourselves to values $\kappa \ll 8\omega_0$):

$$\frac{4\delta_{cr}}{\omega_0 h} = \begin{cases} 1 - \dfrac{x}{2} & \text{for} \quad x \ll 1, \\ \dfrac{1}{2x} & \text{for} \quad x \gg 1. \end{cases} \qquad (10.54)$$

This asymptotic behavior for small and large x is shown in Figure 34. A theoretical analysis for intermediate values is difficult, but it is

perfectly clear that the true unknown curve must smoothly join the two curves corresponding to the limiting cases, as indicated by the dashed curve in the figure.

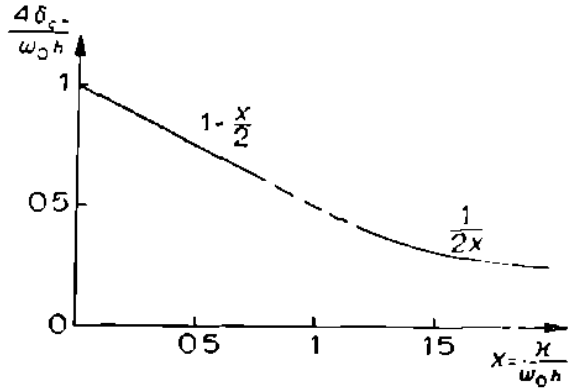

Fig. 34. Dependence of the critical damping on the intensity of the phase diffusion.

The example just considered allows us to study the transition from purely harmonic parametric oscillations to the case of strong fluctuational scatter. Under this transition, the spectral density of the oscillations, initially shaped like a delta-function, gets progressively broader, first reaching a bandwidth of order $\omega_0\langle h\rangle$ and then a much larger bandwidth. The increasing instability makes it progressively more difficult to excite parametric oscillations, i.e., since δ_{cr} decreases monotonically, the values of $\langle\delta\rangle$ leading to parametric excitation must be smaller and smaller.

4. Simultaneous Harmonic and Fluctuational Excitations

Next we consider the case where the parameter changes contain both a purely harmonic component and a relatively broad-band fluctuational component. Then the spectral density of the process $\xi(t)$ has a twofold structure, i.e., it contains a discrete line immersed in a continuous background. Denoting the fluctuational component by $\xi(t)$, we have

$$\xi(t) = h \sin \nu t + \zeta(t). \tag{10.55}$$

The correlation function $\langle \xi \xi_\tau \rangle$ is easily calculated from a knowledge of the characteristic function of the phase increment after time τ:

$$\Theta(v) = \left\langle \exp\left\{ iv \int_t^{t+\tau} \dot{\psi}\, dt \right\} \right\rangle = e^{-\kappa \tau v^2/2} \quad (\tau > 0). \qquad (10.\ 0)$$

If we average over the random initial phase, an oscillatory term depending on t drops out of $\langle \xi \xi_\tau \rangle$, leaving

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$$\langle \xi \xi_\tau \rangle = \frac{h^2}{2} e^{-\kappa|\tau|/2} \cos 2\omega_0 \tau,$$

with corresponding spectral density

$$\kappa(\omega) = \frac{h^2 \kappa}{4} \left[\frac{1}{(\frac{1}{2}\kappa)^2 + (\omega - 2\omega_0)^2} + \frac{1}{(\frac{1}{2}\kappa)^2 + (\omega + 2\omega_0)^2} \right]. \qquad (10.52)$$

Substituting (10.52) into (10.44), we find the critical damping

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It is interesting to compare this result with the critical damping coefficient (10.33) for the case of weak phase diffusion. According to (10.33) and (10.53), we have the following dependence of δ_{cr} on $x = \kappa/\omega_0 h$ (restricting ourselves to values $\kappa \ll 8\omega_0$):

$$\frac{4\delta_{cr}}{\omega_0 h} = \begin{cases} 1 - \dfrac{x}{2} & \text{for} \quad x \ll 1, \\ \dfrac{1}{2x} & \text{for} \quad x \gg 1. \end{cases} \qquad (10.54)$$

This asymptotic behavior for small and large x is shown in Figure 34. A theoretical analysis for intermediate values is difficult, but it is

perfectly clear that the true unknown curve must smoothly join the two curves corresponding to the limiting cases, as indicated by the dashed curve in the figure.

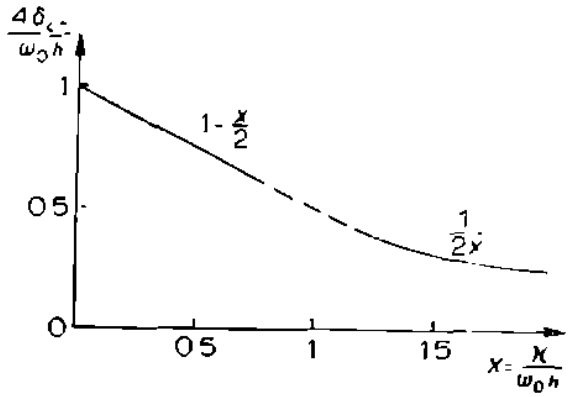

Fig. 34. Dependence of the critical damping on the intensity of the phase diffusion.

The example just considered allows us to study the transition from purely harmonic parametric oscillations to the case of strong fluctuational scatter. Under this transition, the spectral density of the oscillations, initially shaped like a delta-function, gets progressively broader, first reaching a bandwidth of order $\omega_0\langle h\rangle$ and then a much larger bandwidth. The increasing instability makes it progressively more difficult to excite parametric oscillations, i.e., since δ_{cr} decreases monotonically, the values of $\langle\delta\rangle$ leading to parametric excitation must be smaller and smaller.

4. Simultaneous Harmonic and Fluctuational Excitations

Next we consider the case where the parameter changes contain both a purely harmonic component and a relatively broad-band fluctuational component. Then the spectral density of the process $\xi(t)$ has a twofold structure, i.e., it contains a discrete line immersed in a continuous background. Denoting the fluctuational component by $\xi(t)$, we have

$$\xi(t) = h \sin \nu t + \zeta(t). \tag{10.55}$$

Since we are interested in the fundamental parametric resonance, which is described by the first approximation in the standard equations, we shall assume that the frequency ν of the harmonic component is near $2\omega_0$. Thus, introducing the "mismatch"

$$\Delta = 2\omega_0 - \nu, \tag{10.56}$$

we impose the condition

$$\Delta \lesssim \omega_0 h \ll \omega_0 . \tag{10.57}$$

In (10.55) we omit the arbitrary initial phase of the oscillations, which depends on the choice of the time origin.

The derivation of the standard equations is almost the same as in Sec. 1, except that we now define the phase somewhat differently by writing

$$x = A\cos\left(\frac{\nu}{2}t + \varphi\right), \quad \dot{x} = -\frac{\nu}{2}A\sin\left(\frac{\nu}{2}t + \varphi\right) \tag{10.58}$$

instead of (10.7). This leads to the equations

$$\begin{aligned} \dot{u} &= -\delta(t) + \frac{1}{\nu}\left(\omega_0^2 - \frac{\nu^2}{4} + \omega_0^2\xi\right)\sin(\nu t + 2\varphi), \\ \dot{\varphi} &= \frac{1}{\nu}\left(\omega_0^2 - \frac{\nu^2}{4} + \omega_0^2\xi\right)[1 + \cos(\nu t + 2\varphi)] \end{aligned} \tag{10.59}$$

instead of (10.12). Substituting from (10.55) and dropping oscillatory terms, we find that

$$\begin{aligned} \dot{u} &= -\delta(t) + \frac{\omega_0 h}{4}\cos 2\varphi + \frac{\omega_0}{2}\zeta(t)\sin(\nu t + 2\varphi), \\ \dot{\varphi} &= \frac{\Delta}{2} - \frac{\omega_0 h}{4}\sin 2\varphi + \frac{\omega_0}{2}\zeta(t)[1 + \cos(\nu t + 2\varphi)]. \end{aligned} \tag{10.60}$$

Here we have simplified the coefficients by using the inequality $\Delta \ll \omega_0$, which justifies the approximations

$$\omega_0^2 - \frac{\nu^2}{4} \approx \omega_0\Delta, \quad \frac{\omega_0^2}{\nu} \approx \frac{\omega_0}{2}.$$

We shall assume that the fluctuations $\xi(t)$ are comparatively broad-band, i.e., that their correlation time is small compared to the phase

relaxation time

$$\tau_{cor} \ll \frac{1}{\omega_0 h}. \tag{10.61}$$

Then we can apply the stochastic approach to the equations (10.60), just as in Sec. 3. The fluctuational terms in (10.60) are exactly the same as in (10.12), if $\xi(t)$ is replaced by $\zeta(t)$. Suppose we separate each of these terms into two parts, an average value and a fluctuational increment:

$$\begin{aligned} &\frac{\omega_0}{2}\zeta(t)\sin(\nu t + 2\varphi) = m_1 + \zeta_1(t), \\ &\frac{\omega_0}{2}\zeta(t)[1 + \cos(\nu t + 2\varphi)] = m_2 + \zeta_2(t). \end{aligned} \tag{10.62}$$

Then, if we write $\zeta_1 \equiv \xi_1(t) - [\delta(t) - \langle\delta(t)\rangle]$, the equations (10.60) take the form

$$\begin{aligned} \dot{u} &= m_1 - \langle\delta(t)\rangle + \frac{\omega_0 h}{4}\cos 2\varphi + \zeta_1', \\ \dot{\varphi} &= \frac{\Delta}{2} + m_2 - \frac{\omega_0 h}{4}\sin 2\varphi + \zeta_2, \end{aligned} \tag{10.63}$$

where the mean values m_1, m_2 and the intensity coefficients K_1, K_2 are defined as in Sec. 3, after replacing $\xi(t)$ by $\zeta(t)$, for example in (10.37)–(10.44) and (10.46). The most important formulas are

$$\begin{aligned} K_1 &= \frac{\omega_0^2}{8}\kappa(2\omega_0) + \int_{-\infty}^{\infty}[\langle\delta\delta_\tau\rangle - \langle\delta\rangle^2]\,d\tau, \\ K_2 &= \frac{\omega_0^2}{4}[\kappa(0) + \tfrac{1}{2}\kappa(2\omega_0)], \\ m_1 &= \frac{\omega_0}{2}\langle\zeta(t)\sin(\nu t + 2\varphi)\rangle = \frac{\omega_0^2}{8}\kappa(2\omega_0), \end{aligned} \tag{10.64}$$

where $\kappa(\omega)$ is now the spectral density of the random process $\zeta(t)$.

In the case where the damping is nonrandom and constant in time and $\zeta(t)$ contains no low-frequency components, the integral in (10.64) disappears and $\kappa(0)=0$, so that $\zeta_1' = \zeta_1$ and both processes ζ_1 and ζ_2

have the same intensity coefficient

$$K_1 = K_2 = K = \frac{\omega_0^2}{8}\kappa(2\omega_0). \tag{10.65}$$

For simplicity, we shall assume that this is the case and also that $m_2 = 0$, although removing these restrictions does not cause any trouble (for example, a nonzero value of m_2 can always be included in the term $\Delta/2$). Averaging the first of the equations (10.63), we obtain the critical damping

$$\delta_{cr} = m_1 + \frac{\omega_0 h}{4}\langle\cos 2\varphi\rangle. \tag{10.66}$$

Thus, to find δ_{cr} we must calculate $\langle\cos 2\varphi\rangle$, using the second of the equations (10.63).

If the condition (10.61) holds, we can regard the processes ζ_1 and ζ_2 in (10.63) as delta-correlated, with correlation function $K\delta(\tau)$. Then the phase equation gives rise to the Fokker-Planck equation

$$\dot{w}(\varphi) = -\frac{\partial}{\partial\varphi}\left[\left(\frac{\Delta}{2} - \frac{\omega_0 h}{4}\sin 2\varphi\right)w(\varphi)\right] + \frac{K}{2}\frac{\partial^2 w}{\partial\varphi^2}, \tag{10.67}$$

which, except for notation, is the same as equation (9.44), encountered in the theory of synchronization. The stationary solution of this equation was found in Chap. 9. Using the periodicity condition and applying the method which led to the expression (9.52), we obtain the stationary solution of (10.67):

$$w(\varphi) = \frac{1}{N}\exp\left\{\frac{1}{K}\left(\Delta\varphi + \frac{\omega_0 h}{4}\cos 2\varphi\right)\right\} \times \int_{\varphi}^{\varphi+\pi} \exp\left\{-\frac{1}{K}\left(\Delta\psi + \frac{\omega_0 h}{4}\cos 2\psi\right)\right\} d\psi. \tag{10.68}$$

This stationary probability density is periodic with period π. The constant N is determined from the normalization condition

$$\int_0^{2\pi} w(\varphi)\,d\varphi = 1, \tag{10.69}$$

which gives

$$N = 2\int_0^{\pi} d\varphi \int_{\varphi}^{\varphi+\pi} \exp\left\{\frac{1}{K}\left(\Delta\varphi - \Delta\psi + \frac{\omega_0 h}{4}\cos 2\varphi - \frac{\omega_0 h}{4}\cos 2\psi\right)\right\} d\psi,$$

or

$$N = 2\int_0^{\pi}\int_0^{\pi} \exp\left\{-\frac{\Delta}{K}\chi + \frac{\omega_0 h}{2K}\sin\chi\sin(2\varphi+\chi)\right\} d\varphi\, d\chi, \qquad (10.70)$$

after introducing the variable $\chi = \varphi - \psi$. Expressing the integral with respect to φ in terms of a Bessel function and then making the change of variable

$$y = \begin{cases} \dfrac{\pi}{2} - \chi & \text{for} \quad 0 < \chi < \dfrac{\pi}{2}, \\ \chi - \dfrac{\pi}{2} & \text{for} \quad \dfrac{\pi}{2} < \chi < \pi, \end{cases}$$

we obtain

$$N = 2\pi\int_0^{\pi} e^{-\Delta\chi/K} I_0\left(\frac{\omega_0 h}{2K}\sin\chi\right) d\chi = 2\pi^2 e^{-\pi q}\left|I_{iq}\left(\frac{\omega_0 h}{4K}\right)\right|^2 \qquad (10.71)$$

where $q = \Delta/2K$, which is the analogue of (9.53).

To calculate the stationary mean value $\langle\cos 2\varphi\rangle$, we must now average with the weight (10.68). Writing

$$\cos 2\varphi = \cos(2\varphi + \chi - \chi) = \cos\chi\cos(2\varphi+\chi) + \sin\chi\sin(2\varphi+\chi),$$

and replacing χ by $\varphi+\chi$ in (10.68), we have

$$\begin{aligned}\langle\cos 2\varphi\rangle &= 2\int_0^{\pi}\cos 2\varphi\, w(\varphi)\, d\varphi \\ &= \frac{2}{N}\int_0^{\pi}\int_0^{\pi}[\cos\chi\cos(2\varphi+\chi) + \sin\chi\sin(2\varphi+\chi)] \\ &\quad\times\exp\left\{-\frac{\Delta}{K}\chi + \frac{\omega_0 h}{2K}\sin\chi\sin(2\varphi+\chi)\right\} d\varphi\, d\chi. \qquad (10.72)\end{aligned}$$

Because of the way $\cos(2\varphi+\chi)$ and $\sin(2\varphi+\chi)$ change sign, it is easily verified that

$$\int_0^{\pi} \cos(2\varphi+\chi)\exp\left\{\frac{\omega_0 h}{2K}\sin\chi\sin(2\varphi+\chi)\right\}d\varphi = 0$$

for any value of χ. Therefore only the second term in (10.72) has to be taken into account:

$$\langle\cos 2\varphi\rangle = \frac{2}{N}\int_0^{\pi}\int_0^{\pi}\sin\chi\sin(2\varphi+\chi) \times \exp\left\{-\frac{\Delta}{K}\chi+\frac{\omega_0 h}{2K}\sin\chi\sin(2\varphi+\chi)\right\}d\varphi\,d\chi. \tag{10.73}$$

But the integral (10.73) can be obtained from (10.70) by differentiating with respect to $\omega_0 h/2K$, and hence the desired mean value can be simply expressed in terms of the function (10.71), i.e.,

$$\langle\cos 2\varphi\rangle = \frac{1}{N}\frac{dN}{d\left(\frac{\omega_0 h}{2K}\right)} = F\left(\frac{\omega_0 h}{4K}, \frac{\Delta}{2K}\right), \tag{10.74}$$

where

$$F(z,q) = \frac{1}{2}\frac{d}{dz}\ln N(z) = \frac{1}{2}\frac{d}{dz}\ln\left[I_{iq}(z)I_{-iq}(z)\right]. \tag{10.75}$$

Using elementary properties of the function $I_p(z)$,[2] we can also write (10.75) in the form

$$F(z,q) = \frac{1}{2}\left[\frac{I_{1+iq}(z)}{I_{iq}(z)} + \frac{I_{1-iq}(z)}{I_{-iq}(z)}\right]. \tag{10.76}$$

Substituting (10.74) into (10.66) and dividing both sides of the

[2] We have in mind the recurrence relations

$$I_{p-1}(z) + I_{p+1}(z) = 2I_p'(z),$$

$$I_{p-1}(z) - I_{p+1}(z) = \frac{2p}{z}I_p(z).$$

resulting formula by

$$K = \frac{\omega_0^2}{8}\kappa(2\omega_0),$$

we can express the critical damping in the form

$$\frac{\delta_{cr}}{K} = 1 + \frac{\omega_0 h}{4K} F\left(\frac{\omega_0 h}{4K}, \frac{\Delta}{2K}\right). \tag{10.77}$$

Using (10.75) or (10.76), we can now obtain numerical results by using tables of Bessel functions of imaginary argument and imaginary order. Moreover, in various special cases we can deduce information by using different asymptotic expressions for the cylinder functions. In this regard, the important thing is the relation between the arguments $\omega_0 h/4K$, $\Delta/2K$ and unity. Depending on the size of $\omega_0 h/4K$, we have three cases, i.e., relatively large amplitudes of the harmonic excitation ($\omega_0 h/4K \gg 1$), intermediate amplitudes ($\omega_0 h/4K \sim 1$) and relatively strong noise ($\omega_0 h/4K \ll 1$). But at the same time we can independently use the size of $\Delta/2K$ to distinguish the cases of small mismatch ($\Delta/2K \ll 1$), intermediate mismatch ($\Delta/2K \sim 1$) and large mismatch ($\Delta/2K \gg 1$). In Table 3 we classify all these different cases, which lead to 11 distinct regions (the ninth region can be further subdivided into 3 subregions).

TABLE 3

	$\frac{\Delta}{2K} \ll 1$	$\frac{\Delta}{2K} \sim 1$	$\frac{\Delta}{2K} \gg 1$	
$\frac{\omega_0 h}{4K} \gg 1$	1. $\frac{2\Delta}{\omega_0 h} \ll 1$	2. $\frac{2\Delta}{\omega_0 h} \ll 1$	3. $\frac{2\Delta}{\omega_0 h} \ll 1$	4. $\frac{2\Delta}{\omega_0 h} \sim 1$ | 5. $\frac{2\Delta}{\omega_0 h} \gg 1$
$\frac{\omega_0 h}{4K} \sim 1$	6. $\frac{2\Delta}{\omega_0 h} \ll 1$	7. $\frac{2\Delta}{\omega_0 h} \sim 1$	8. $\frac{2\Delta}{\omega_0 h} \gg 1$	
$\frac{\omega_0 h}{4K} \ll 1$	9.	10. $\frac{2\Delta}{\omega_0 h} \gg 1$	11. $\frac{2\Delta}{\omega_0 h} \gg 1$	

We now discuss the various special cases, beginning with large amplitudes $\omega_0 h/4K \gg 1$ (regions 1–5). In regions 1–2 we can use the asymptotic representation of the Bessel function

$$J_{\pm iq}(iz) \approx \frac{1}{\sqrt{2\pi z}} e^{z \mp \frac{1}{4}\pi q}. \tag{10.78}$$

For large $q \gg 1$ (in regions 3–5, 8, 11) we can use formula (9.57). Moreover, in region 3 and part of region 4, we can replace the functions $H^{(1)}_{1/3}(iy)$, $H^{(2)}_{1/3}(iy)$, where

$$y = \frac{1}{3q^2}(z^2 - q^2)^{3/2},$$

by asymptotic expressions valuable for large $y \gg 1$. This leads to the formula

$$J_{\pm iq}(iz) \approx \frac{1}{\sqrt{2\pi q v}} e^{q(v - \operatorname{Arc\,tan} v)} \quad \left(v = \frac{1}{q}\sqrt{z^2 - q^2}\right), \tag{10.79}$$

which clearly includes (10.78) as a special case, since whether we choose Arc tan v equal to arc tan v or $-\pi + \text{arc tan } v$ depends on the sign of the index $\pm iq$. Substituting (10.79) into (10.75) and differentiating, we obtain

$$F(z, q) \approx \frac{1}{z}\sqrt{z^2 - q^2} - \frac{1}{2}\frac{z}{z^2 - q^2}. \tag{10.80}$$

Therefore the critical damping (10.77) is given by the approximate formula

$$\frac{\delta_{cr}}{K} = \frac{\omega_0 h}{4K}\sqrt{1 - \left(\frac{2\Delta}{\omega_0 h}\right)^2} + 1 - \frac{1}{2}\frac{1}{\sqrt{1 - \left(\frac{2\Delta}{\omega_0 h}\right)^2}}, \tag{10.81}$$

valid in regions 1–3 and part of region 4.

The main term in (10.81), which by itself gives

$$\delta_{cr} = \tfrac{1}{4}\sqrt{\omega_0^2 h^2 - 4\Delta^2},$$

is not of a fluctuational character, and can be obtained by using a simple theory ignoring fluctuations. On the other hand, the presence of the last two terms is associated with the random excitation. These

terms describe the displacement of the boundary of the region of parametric excitation as a result of fluctuational "impacts."

It is apparent from (10.81) that parametric excitation becomes progressively more difficult as the mismatch (more exactly, the relative mismatch $2\Delta/\omega_0 h$) is increased. The presence of the fluctuations at first enhances excitation, for $(2\Delta/\omega h)^2 < \frac{1}{2}$, and then, for $(2\Delta/\omega_0 h)^2 > \frac{1}{2}$, inhibits the excitation caused by harmonic parametric oscillations.

In region 4, the case of values of $2\Delta/\omega_0 h$ close to unity occupies a special position. Then formula (10.81) ceases to be valid, and moreover, it is just for such values that the contribution of the nonfluctuational first term in (10.81) becomes comparable to the contribution of the other terms. In other words, the role of the harmonic parametric changes in the excitation process decreases to the point where it is comparable to the role of the random parametric fluctuations. For an even larger mismatch

$$\frac{2\Delta}{\omega_0 h} - 1 \sim 1,$$

the harmonic oscillations become less important than the random fluctuations. We shall not analyze the transition region

$$\frac{2\Delta}{\omega_0 h} - 1 \ll 1$$

in detail, since it contributes only a small part of the whole range $2\Delta/\omega_0 h \sim 1$ comprised in region 4. Such an analysis can be made by using formula (9.57) to obtain an expression for δ_{cr}/K involving various cylinder functions of order 1/3 and their derivatives, in fact the logarithmic derivatives of the very combinations of cylinder functions appearing in (9.60) and (9.63).

For values

$$\frac{2\Delta}{\omega_0 h} - 1 \gtrsim 1,$$

i.e., in the rest of region 4 and in regions 5, 8 and 11, the cylinder functions of order 1/3 have arguments greatly exceeding unity and can therefore be replaced by their asymptotic expressions. As a result, we

obtain

$$J_{iq}(iz)J_{-iq}(iz)=\left(\frac{u}{2\sqrt{3}}\right)^2\left|H^{(1)}_{1/3}\left(\frac{qu^3}{3}\right)\right|^2\approx\frac{1}{2\pi qu}\qquad(qu=\sqrt{q^2-z^2}),$$

and hence (10.82)

$$F(z,q)=\frac{1}{2}\frac{z}{\sqrt{q^2-z^2}},$$

$$\frac{\delta_{cr}}{K}=1+\frac{1}{2}\frac{1}{\left(\frac{2\Delta}{\omega_0 h}\right)^2-1} \tag{10.83}$$

in regions 5, 8, 11 and part of region 4.

The only difference between (10.83) and (10.81) is that the first term of (10.81) is absent. In the intermediate region $(2\Delta/\omega_0 h)-1\ll 1$, the cylinder functions smoothly ("analytically") cause this term to disappear. The number 1 appearing in the right-hand side of (10.83) has a simple meaning, i.e., it is the value of the ratio δ_{cr}/K in the absence of harmonic parametric oscillations. The effect of these oscillations is described by the second term

$$\frac{1}{2}\frac{1}{\left(\frac{2\Delta}{\omega_0 h}\right)^2-1}.$$

Thus the harmonic oscillations increase the extent to which the fluctuations are effective in producing excitation. However, even this stimulating effect drops to zero as the amplitude h is decreased (or the mismatch Δ increased), and hence it can be neglected in regions 5, 8 and 11.

The dominant role played by the harmonic parametric oscillations in region 1 disappears if we increase the mismatch or noise level, or decrease the amplitude h, i.e., as region 1 turns into region 6 or 9. We now examine this phenomenon in more detail, for the case of zero mismatch, which corresponds to the most favorable condition for excitation of parametric oscillations. Setting $\Delta=0$, $q=0$ in (10.75) or (10.76), we find that

$$F(z,0)=\frac{I_1(z)}{I_0(z)}, \tag{10.84}$$

so that formula (10.77) becomes

$$\frac{\delta_{cr}}{K} = 1 + \frac{\omega_0 h}{4K} \frac{I_1\left(\frac{\omega_0 h}{4K}\right)}{I_0\left(\frac{\omega_0 h}{4K}\right)} \tag{10.85}$$

in regions 1, 6 and 9. In region 1, the second term predominates. In this case, the asymptotic formulas imply

$$\frac{I_1(z)}{I_0(z)} = 1 - \frac{1}{2z} + O\left(\frac{1}{z^2}\right).$$

This allows us to separate (10.85) into a basic nonfluctuational term and a main noise correction, whose effect is comparatively small:

$$\frac{\delta_{cr}}{K} = \frac{\omega_0 h}{4K} + \frac{1}{2} \quad \text{(in region 1)}. \tag{10.86}$$

As the amplitude of the harmonic oscillations is decreased (or the noise level increased), the second term in the right-hand side of (10.85) decreases and becomes comparable to the first purely fluctuational term. As the amplitude is decreased further, it begins to play a subordinate role, and in region 9 the basic correction due to the harmonic oscillations can be written in the simpler form

$$\frac{\delta_{cr}}{K} = 1 + \frac{1}{2}\left(\frac{\omega_0 h}{4K}\right)^2. \tag{10.87}$$

The dependence of the critical damping (10.85) on the ratio $\omega_0 h/4K$ is shown in Figure 35. Both kinds of parametric oscillations, harmonic and fluctuational, lead to parametric excitation.

In regions 9 and 10, as in regions 5, 8 and 11, the role of harmonic parametric oscillations is very small. In fact, the function $F = \langle \cos 2\varphi \rangle$ cannot exceed unity, while the factor $\omega_0 h/4K$ is small and makes the second term in (10.77) much smaller than the purely fluctuational first term. It should be noted that the elementary theory given in Sec. 3 is sufficient in regions 5, 8 and 11. Region 7 is the most difficult to analyze, since simple asymptotic expressions are no longer applicable.

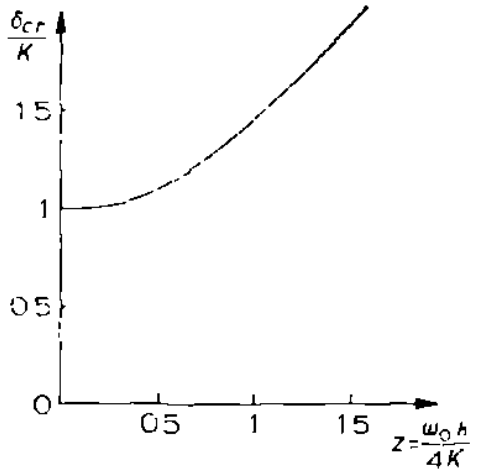

Fig. 35. The critical damping for zero mismatch.

Here we must make direct use of formula (10.75), expressing $F(z,q)$ in terms of Bessel functions of imaginary argument and imaginary order.

5. The Amplitude Distribution. Effect of Nonlinearity

So far we have been interested in the condition $\langle\delta\rangle<\delta_{cr}$ governing parametric excitation. However, the above theory can be made to yield further information, concerning oscillations of amplitude in a parametric system.

Thus we return to the equations (10.12). If we are interested in the behavior of the system during time intervals Δt much larger than the correlation time of the random function $\xi(t)\sin(2\omega_0 t+2\varphi)$ [and also of the function $\delta(t)$, whenever $\delta(t)$ is random], the first equation takes the form

$$\dot{u}=\delta_{cr}-\langle\delta\rangle+\xi_1(t) \tag{10.88}$$

[cf. (10.47)], containing the delta-correlated process of intensity (10.46). Equation (10.88) corresponds to the Fokker-Planck equation (10.45). If the quantity u is fixed at time $t=0$, i.e., $u(0)=u_0$, then the probability density $w(u,t)$ at any other instant of time is obtained by solving (10.45). As is easily verified, the solution of (10.45)

is given by

$$w(u,t)=\frac{1}{\sqrt{2\pi K_1 t}}\exp\left\{-\frac{1}{2K_1 t}\left[u-u_0-(\delta_{cr}-\langle\delta\rangle)t\right]^2\right\}. \tag{10.89}$$

By making the simple change of variable $u=\ln A$, we go from the logarithm of the (physical) amplitude of the oscillations to the amplitude itself. This gives the following law for the evolution in time of the amplitude distribution of a linear parametric system:

$$w(A,t)=\frac{1}{\sqrt{2\pi K_1 t}}A^{-1}\exp\left\{-\frac{1}{2K_1 t}\ln^2\left(\frac{A}{A_0}e^{-\gamma t}\right)\right\} \tag{10.90}$$
$$(\gamma=\delta_{cr}-\langle\delta\rangle,\ A_0=e^{u_0}).$$

Given any amplitude level b, it is not hard to see that as $t\to\infty$ the probability of the inequality $A(t)>b$ approaches 1 if $\delta_{cr}-\langle\delta\rangle>0$ and 0 if $\delta_{cr}-\langle\delta\rangle<0$. Therefore a stationary regime of parametric oscillations is impossible in a purely linear system, since the oscillations would either increase without limit or fall off to zero. Of course, in any practical system the amplitude cannot become arbitrarily large, since the system would be destroyed unless some nonlinearity limiting the amplitude came into play. Thus we must introduce nonlinearity before the stationary amplitude distribution can be calculated.

As an example of a nonlinear parametric system, consider the self-excited system described by the equation

$$\ddot{y}+2\left[\delta+\alpha y^2\right]\dot{y}+\omega_0^2\left[1+\xi(t)\right]y=0\quad(\alpha>0), \tag{10.91}$$

where, compared to (10.1), there is a nonlinear term, the damping δ is taken to be constant, and $k(t)$ is chosen in the form $\omega_0^2\left[1+\xi(t)\right]$, instead of (10.5). This time we do not make the transformation (10.3), so that $x=y$ and $A=e^u$ is the physical amplitude, rather than the amplitude of the fictitious process (10.3). Going over to the variables u and φ defined by the equations

$$y=e^u\cos(\omega_0 t+\varphi),\quad \dot{y}=-\omega_0 e^u\sin(\omega_0 t+\varphi), \tag{10.92}$$

we obtain the fluctuation equations

$$\begin{aligned} \dot{u} &= \delta_{cr} - \delta - \alpha e^{2u} + \xi_1(t), \\ \dot{\varphi} &= m_2 + \xi_2(t), \end{aligned} \tag{10.93}$$

by the usual method [cf. (10.47)]. Assuming that the process $\xi(t)$ has a sufficiently small correlation time satisfying the conditions

$$\tau_{cor} \ll \frac{1}{|\delta_{cr} - \delta|}, \quad \tau_{cor} \ll T_{rel},$$

we can regard the random processes ξ_1, ξ_2 as delta-correlated, i.e.,

$$\langle \xi_1 \xi_{1\tau} \rangle = K_1 \delta(\tau), \quad \langle \xi_2 \xi_{2\tau} \rangle = K_2 \delta(\tau),$$

where the quantities δ_{cr}, m_2, K_1 and K_2 are given by the previous formulas (10.40), (10.42), (10.44) and (10.46).[3]

The Fokker-Planck equation corresponding to the first of the equations (10.93) is

$$\dot{w}(u) = -\frac{\partial}{\partial u}\left[(\delta_{cr} - \delta - \alpha e^{2u}) w(u)\right] + \frac{K_1}{2} \frac{\partial^2 w(u)}{\partial u^2}, \tag{10.94}$$

with stationary solution

$$w(u) = C \exp\left\{\frac{2}{K_1}(\delta_{cr} - \delta)u - \frac{\alpha}{K_1} e^{2u}\right\}. \tag{10.95}$$

Returning to the amplitude $A = e^u$ and determining the normalization constant C, we obtain the stationary amplitude distribution

$$w(A) = \frac{2}{\Gamma\left(1 - \frac{\delta}{K_1}\right)} \left(\frac{\alpha}{K_1}\right)^{1-(\delta/K_1)} A^{1-2(\delta/K_1)} e^{-\alpha A^2/K_1} \tag{10.96}$$

(after using the formula $\delta_{cr} = K_1$). Except for notation, this is the same as the probability density (7.43), plotted in Figure 20, p. 183. In the case where $\delta \ll K_1$, (10.96) is close to being a Rayleigh distribution.

The expression (10.96) is meaningful only if the condition $K_1 > \delta$ for parametric excitation is satisfied. In the opposite case, (10.96) has a

[3] Note that now

$$K_1 = \delta_{cr} = \frac{\omega_0^2}{8} \kappa(2\omega_0).$$

nonintegrable singularity at zero ($A=0$), which corresponds to taking $w(A)$ to be a delta function $\delta(A)$. In reality, of course, fluctuations of a nonparametric nature prevent the amplitude from vanishing exactly, even when the parametric system is stable. In fact, these fluctuations appear as an additive term in (10.91), producing a certain minimum amount of amplitude scatter even under stable conditions. As a result of these two effects, the limiting of the amplitude by nonlinearity in the system in the case of parametric instability and the presence of a minimum amount of amplitude scatter under stable conditions, there is no qualitative change in behavior in going from stable to unstable conditions (as would occur in an ideal linear system). This transition is accompanied by a certain broadening of the amplitude distribution, a broadening which is more noticeable the smaller the additive fluctuations. If the nonlinearity and the level of additive fluctuations are both large, the broadening of the probability density $w(A)$ due to loss of stability can hardly be distinguished from the broadening due to increasing K_1 within the region of applicability of formula (10.96).

6. Parametric Systems with Two Branches. Parametric Amplifiers

Another phenomenon of interest in radio engineering (besides parametric excitation) is that of *parametric amplification*, which occurs when an external excitation, described by an additive term in the equation for the oscillations, is applied to a parametric system in the stable state. Then the "pumping action" of the energy of the parametric oscillations, although not large enough to cause excitation, "accumulates" and is transferred to the signal, causing it to be amplified.

In the case of harmonic excitation, the equation describing the parametric system takes the form[4]

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2[1 + h\sin(\nu t + \psi)]x = -\omega_0^2 E\sin(\omega_s t + \gamma). \tag{10.97}$$

[4] Here again we do not make the transformation (10.3), so that $x = y$ and A is the physical amplitude. For the time being, we ignore random fluctuations in the parametric oscillations $h\sin(\nu t + \psi)$.

Suppose the external frequency ω_s is close to the natural frequency ω_0 and ν is close to $2\omega_0$. Then, in the usual way, the transformation (10.7) leads to the following truncated equations:

$$\dot{A} = -\delta A + \frac{\omega_0 h}{4} A \cos[(2\omega_0 - \nu)t + 2\varphi - \psi] + \frac{\omega_0 E}{2} \cos[(\omega_0 - \omega_s)t + \varphi - \chi],$$

$$\dot{\varphi} = -\frac{\omega_0 h}{4} \sin[(2\omega_0 - \nu)t + 2\varphi - \psi] - \frac{\omega_0 E}{2A} \sin[(\omega_0 - \omega_s)t + \varphi - \chi].$$

If both "mismatches" vanish, i.e., if $\omega_0 - \omega_s = 0$, $2\omega_0 - \nu = 0$, these equations become

$$\begin{aligned} \dot{A} &= -\delta A + \frac{\omega_0 h}{4} A \cos(2\varphi - \psi) + \frac{\omega_0 E}{2} \cos(\varphi - \psi), \\ \dot{\varphi} &= -\frac{\omega_0 h}{4} \sin(2\varphi - \psi) - \frac{\omega_0 E}{2A} \sin(\varphi - \psi). \end{aligned} \tag{10.98}$$

It can easily be seen from (10.98) that the amplification depends in an essential way on the phase relation between the parametric oscillations $h \sin(\nu t + \psi)$ and the external oscillations $E \sin(\omega_s t + \chi)$. In fact, according to the second of the equations (10.98), for the "favorable" phase relation $\psi = 2\chi$, a constant phase $\varphi = \chi = \psi/2$ is established, and the first equation takes the form

$$\dot{A} = -\left(\delta - \frac{\omega_0 h}{4}\right) A + \frac{\omega_0 E}{2},$$

corresponding to an amplification factor of

$$\frac{A}{E} = \frac{\omega_0}{2} \frac{1}{\delta - \frac{\omega_0 h}{4}}.$$

However, for an "unfavorable" phase relation, e.g., $\psi = 2\chi \pm \pi$, parametric oscillations not only do not guarantee amplification, but, on the contrary, lead to extra damping in the circuit, with concommitant signal attenuation. Thus, if $E \gg h$, we have $\varphi \approx \chi$, and hence

$$\frac{A}{E} = \frac{\omega_0}{2} \frac{1}{\delta - \frac{\omega_0 h}{4} \cos(2\chi - \psi)},$$

which becomes

$$\frac{A}{E} = \frac{\omega_0}{2} \frac{1}{\delta + \frac{\omega_0 h}{4}}$$

if $2\chi - \psi = \pm\pi$. Although this dependence of amplification on phase is sometimes useful in special applications, it is in general unsuitable for the purposes of undistorted signal amplification, and can be avoided by the use of parametric systems with two branches, to which we now turn our attention.

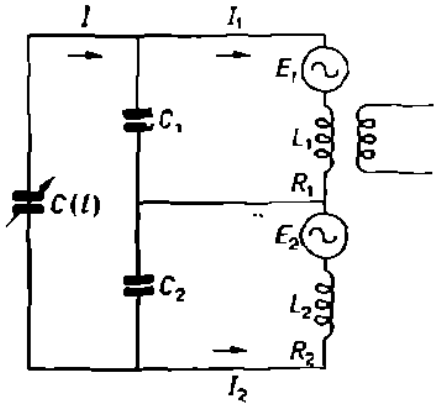

Fig. 36. A parametric system with two branches.

Figure 36 shows a parametric system with two branches, coupled by a variable capacitance $C(t)$. External e.m.f.'s (or e.m.f.'s due to thermal noise in the resistances R_1 and R_2) are applied to each branch, and are denoted by $E_1(t)$ and $E_2(t)$. Using the currents I_1, I_2 and I

shown in the figure, we write down Kirchhoff's equations

$$L_1 I_1 + R_1 I_1 = -\frac{1}{C_1}\int (I_1 - I)\,dt + E_1,$$

$$L_2 I_2 + R_2 I_2 = -\frac{1}{C_2}\int (I_2 + I)\,dt + E_2,$$

$$\frac{1}{C(t)}\int I\,dt + \frac{1}{C_1}\int (I - I_1)\,dt + \frac{1}{C_2}\int (I + I_2)\,dt = 0.$$

In terms of the notation

$$x_1 = \frac{1}{C_1}\int I_1\,dt, \quad x_2 = \frac{1}{C_2}\int I_2\,dt,$$

the last equation gives

$$\left[\frac{1}{C(t)} + \frac{1}{C_1} + \frac{1}{C_2}\right]\int I\,dt = x_1 - x_2.$$

Therefore the first two equations become

$$\ddot{x}_1 + 2\delta_1\dot{x}_1 + \omega_1^2 x_1 = \omega_1^2 \frac{C_1^{-1}}{C^{-1}(t) + C_1^{-1} + C_2^{-1}}(x_1 - x_2) + \omega_1^2 E_1,$$

$$\ddot{x}_2 + 2\delta_2\dot{x}_2 + \omega_2^2 x_2 = \omega_2^2 \frac{C_2^{-1}}{C^{-1}(t) + C_1^{-1} + C_2^{-1}}(x_2 - x_1) + \omega_2^2 E_2,$$

where

$$\omega_k^2 = \frac{1}{L_k C_k}, \quad 2\delta_k = \frac{R_k}{L_k} \quad (k = 1, 2).$$

These equations can also be written in the form

$$\ddot{x}_1 + \omega_1^2 x_1 = \omega_1^2 f_1, \quad \ddot{x}_2 + \omega_2^2 x_2 = \omega_2^2 f_2,$$

by introducing the notation

$$f_1 = -\frac{2\delta_1}{\omega_1^2}\dot{x}_1 + \gamma H(t)(x_1 - x_2) + E_1,$$

$$f_2 = -\frac{2\delta_2}{\omega_2^2}\dot{x}_2 + \frac{1}{\gamma}H(t)(x_2 - x_1) + E_2, \tag{10.99}$$

$$H(t) = \frac{C(t)\sqrt{C_1 C_2}}{C_1 C_2 + (C_1 + C_2)C(t)}, \quad \gamma = \sqrt{\frac{C_2}{C_1}}.$$

It can be assumed that the d-c component of the variable capacitance $C(t)$ is small and hence has a negligible effect on the natural frequencies ω_1 and ω_2 of the branches. Otherwise, instead of x_1 and x_2 themselves, we could consider linear combinations of x_1 and x_2 (corresponding to the "normal modes"), thereby obtaining a circuit equivalent to that shown in Figure 35, but with a small coupling capacitance. On the other hand, we assume that the a-c components of $H(t)$, δ_1, δ_2, E_1 and E_2 are small. Then the processes x_1 and x_2 are narrow-band, and they can be investigated by using the methods described above.

With these assumptions the complex amplitudes

$$z_1 = \left(x_1 + \frac{\dot{x}_1}{i\omega_1}\right) e^{-i\omega_1 t}, \quad z_2 = \left(x_2 + \frac{\dot{x}_2}{i\omega_2}\right) e^{-i\omega_2 t} \tag{10.100}$$

[cf. (I.7.31)] vary only slightly during the periods $2\pi/\omega_1$ and $2\pi/\omega_2$. According to (I.7.32), the time variation of the amplitudes (10.100) is described by the equations

$$\begin{aligned} \dot{z}_1 &= \frac{1}{i\omega_1} e^{-i\omega_1 t}(\ddot{x}_1 + \omega_1^2 x_1) = -i\omega_1 e^{-i\omega_1 t} f_1, \\ \dot{z}_2 &= -i\omega_2 e^{-i\omega_2 t} f_2, \end{aligned} \tag{10.101}$$

where the arguments of the functions f_1 and f_2 are given by

$$\begin{aligned} x_k &= \tfrac{1}{2}(z_k e^{i\omega_k t} + z_k^* e^{-i\omega_k t}), \\ \dot{x}_k &= \frac{i\omega_k}{2}(z_k e^{i\omega_k t} - z_k^* e^{-i\omega_k t}) \qquad (k = 1, 2) \end{aligned} \tag{10.102}$$

[cf. (I.7.34)]. Substituting (10.102) into (10.99), we note that the terms of the form $z_k^* e^{-2i\omega_k t}$ have only a slight effect and can be dropped. In fact, during half a period, z_k^* remains virtually unchanged during half a period, and hence the effects of the term $z_k^* e^{-2i\omega_k t}$ cancel each other out. As a result, we obtain

$$\begin{aligned} \dot{z}_1 + \delta_1 z_1 = -\frac{i\omega_1 \gamma}{2}(Hz_1 + He^{-2i\omega_1 t} z_1^* - He^{i(\omega_2 - \omega_1)t} z_2 \\ - He^{-i(\omega_1 + \omega_2)t} z_2^*) - i\omega_1 E_1 e^{-i\omega_1 t}, \\ \dot{z}_2 + \delta_2 z_2 = -\frac{i\omega_2}{2\gamma}(Hz_2 + He^{-2i\omega_2 t} z_2^* - He^{i(\omega_1 - \omega_2)t} z_1 \\ - He^{-i(\omega_1 + \omega_2)t} z_1^*) - i\omega_2 E_2 e^{-i\omega_2 t}. \end{aligned} \tag{10.103}$$

It is clear from (10.103) that the complex amplitude z_1 (say) is appreciably affected by the spectral components of the function $H(t)$ near the frequencies $0, 2\omega_1, |\omega_1-\omega_2|, \omega_1+\omega_2$, and also by the components of $E_1(t)$ near the frequency ω_1. The first frequency band is due to instability of the natural frequency of the branch due to slow parametric fluctuations, while the second frequency band $\omega\approx 2\omega_1$ corresponds to the fundamental parametric resonance. Both of these phenomena are covered by the ordinary theory of parametric oscillations in a single-branch system, and hence will not be considered here. Instead we restrict ourselves to an investigation of the band $\omega\approx\omega_1+\omega_2$, due entirely to interaction between the two branches.

6.1. Suppose $H(t)$ and $E_1(t)$ are close to being harmonic oscillations of frequencies ω_1 and ω_2, respectively, i.e.,

$$\begin{aligned} H(t) &= h_0(t)\sin[(\omega_1+\omega_2)t+\psi(t)], \\ E_1(t) &= -E_0(t)\sin[\omega_1 t+\chi(t)], \quad E_2(t)=0, \end{aligned} \tag{10.104}$$

where $h_0(t)$, $E_0(t)$, $\psi(t)$ and $\chi(t)$ are slowly varying amplitudes and phases. Writing sines in the form

$$\sin\Phi=\frac{1}{2i}(e^{i\Phi}-e^{-i\Phi}),$$

and substituting (10.104) into (10.103), we find that most of the terms on the right are rapidly varying and can be dropped. In fact, the only terms that remain are

$$\begin{aligned} \dot z_1+\delta_1 z_1 &= \frac{\omega_1\gamma}{4}h(t)z_2^*+\frac{\omega_1}{2}E(t), \\ \dot z_2+\delta_2 z_2 &= \frac{\omega_2}{4\gamma}h(t)z_1^*, \end{aligned} \tag{10.105}$$

where

$$h(t)=h_0(t)\,e^{i\psi(t)}, \quad E(t)=E_0(t)\,e^{i\chi(t)}$$

are the complex signal amplitudes.

In the special case where the signals $H(t)$ and $E_1(t)$ are strictly harmonic with frequencies ν and ω_s, the equations (10.105) take the

form

$$\begin{aligned} \dot{z}_1 + \delta_1 z_1 &= \frac{\omega_1 \gamma}{4} h e^{i\Delta t} z_2^* + \frac{\omega_1}{2} E e^{i\Delta_1 t}, \\ \dot{z}_2^* + \delta_2 z_2^* &= \frac{\omega_2}{4\gamma} h^* e^{-i\Delta t} z_1, \end{aligned} \tag{10.106}$$

where $h = h_0 e^{i\psi_0}$, $E = E_0 e^{i\chi_0}$ and Δ_1, Δ denote the mismatches

$$\Delta_1 = \omega_s - \omega_0, \quad \Delta = \nu - \omega_1 - \omega_2.$$

Then the change of variables

$$z_1' = e^{-\Delta_1 t} z_1, \quad z_2' = e^{-\Delta_2 t} z_2 \qquad (\Delta_2 = \Delta - \Delta_1)$$

transforms (10.106) into the equations

$$\begin{aligned} \dot{z}_1' + \alpha_1 z_1' - \frac{\omega_1 \gamma}{4} h z_2'^* &= \frac{\omega_1}{2} E, \\ \dot{z}_2'^* + \alpha_2^* z_2'^* - \frac{\omega_2}{4\gamma} h^* z_1' &= 0 \end{aligned} \tag{10.107}$$

with constant coefficients, where

$$\alpha_1 = \alpha_1(\Delta_1) = \delta_1 + i\Delta_1, \quad \alpha_2 = \alpha_2(\Delta_2) = \delta_2 + i\Delta_2. \tag{10.108}$$

Looking for a solution of the system (10.107) in the form const e^{pt}, we obtain the characteristic equation

$$p^2 + (\alpha_1 + \alpha_2^*) p + \alpha_1 \alpha_2^* - \frac{\omega_1 \omega_2}{16} h_0^2 = 0$$

determining p, with solution

$$2p_{1,2} = -\alpha_1 - \alpha_2^* \pm \sqrt{(\alpha_1 - \alpha_2^*)^2 + \frac{\omega_1 \omega_2}{4} h_0^2}.$$

For $h_0 = 0$, we have $p_1 = -\alpha_1, p_2 = -\alpha_2^*$. As h_0 increases, the damping in the system decreases, with the absolute value of the real part of one of the roots p_1, p_2 becoming less that δ_1, δ_2. For sufficiently large $h_0 = h_{cr}$, this real part vanishes, and for still larger values $h_0 > h_{cr}$, it becomes positive, testifying to unlimited growth of the oscillations, i.e., self-excited oscillations. From the condition Re $p = 0$, we find the critical value of the amplitude of the "driving force," corresponding

to loss of stability:

$$h_{cr} = 4\sqrt{\frac{\delta_1\delta_2}{\omega_1\omega_2}}\sqrt{1+\left(\frac{\Delta}{\delta_1+\delta_2}\right)^2}. \tag{10.109}$$

Parametric amplification can only occur in the stable regime, when $\operatorname{Re} p_1<0$, $\operatorname{Re} p_2<0$ and all the transients are damped out with time constants

$$T_{1,2} = \frac{1}{-\operatorname{Re} p_{1,2}}.$$

In the case where the frequency of the driving force is tuned to equal the sum of the natural frequencies $\nu=\omega_1+\omega_2$, we have $\Delta=0$, and hence

$$T_{1,2} = \frac{2}{\delta_1+\delta_2 \mp \sqrt{(\delta_1-\delta_2)^2+\frac{1}{4}\omega_1\omega_2 h_0^2}}.$$

When an amount of time $t \gg T_{1,2}$ has elapsed from the initial time, all the transients die out and a stable regime is established, corresponding to zero derivatives $\dot{z}_1'$, $\dot{z}_2'^*$ in the equations (10.107), which can then be solved for the stationary complex amplitudes of the forced oscillations:

$$\begin{aligned} z_1' &= z_1 e^{-i\Delta_1 t} = \frac{\omega_1}{2}\frac{\alpha_2^*}{\alpha_1\alpha_2^* - \frac{1}{16}\omega_1\omega_2 h_0^2}E, \\ z_2' &= z_2 e^{-i\Delta_2 t} = \frac{1}{8\gamma}\frac{\omega_1\omega_2 h}{\alpha_1^*\alpha_2 - \frac{1}{16}\omega_1\omega_2 h_0^2}E^*. \end{aligned} \tag{10.110}$$

To find the amplification factor, we take the absolute value of z_1'/E, obtaining

$$k_{11}(\omega_s) \equiv \frac{A_1}{E_0} = \left|\frac{z_1'}{E}\right| = \frac{\omega_1}{2}\left|\frac{\alpha_2^*}{\alpha_1\alpha_2^* - \frac{1}{16}\omega_1\omega_2 h_0^2}\right|.$$

Using the definition (10.108) of the quantities α_1 and α_2, we find the following expression for the square of the amplification factor:

$$\begin{aligned} k_{11}^2(\omega_s) &= \frac{\omega_1^2}{4}\left|\frac{\alpha_2(\omega_s-\omega_1-\Delta)}{\alpha_1(\omega_s-\omega_1)\alpha_2(\omega_s-\omega_1-\Delta) - \frac{1}{16}\omega_1\omega_2 h_0^2}\right|^2 \\ &= \frac{\omega_1^2}{4}\frac{\delta_2^2+(\omega_s-\omega_1-\Delta)^2}{D}, \end{aligned}$$

$$D = [\delta_1\delta_2 - \tfrac{1}{16}\omega_1\omega_2 h_0^2 - (\omega_s-\omega_1)(\omega_s-\omega_1-\Delta)]^2 + [\delta_1(\omega_s-\omega_1-\Delta) + \delta_2(\omega_s-\omega_1)]^2. \tag{10.111}$$

Similarly, we obtain

$$k_{21}^2(\omega_s) \equiv \frac{A_2^2}{E_0^2} = \left|\frac{z_2'}{E}\right|^2 = \frac{1}{64\gamma^2}\,\frac{\omega_1^2\omega_2^2 h_0^2}{|\alpha_1(\omega_s-\omega_1)\alpha_2(\omega_s-\omega_1-\Delta) - \tfrac{1}{16}\omega_1\omega_2 h_0^2|^2}. \tag{10.112}$$

In the same way, we can calculate the amplification factor for the case where a harmonic voltage $E_2(t) = -E_0 \sin(\omega_s t + \chi_0)$ is applied to the second branch, by simply permuting the indices 1 and 2 in (10.111) and (10.112):

$$k_{12}^2(\omega_s) = \frac{\gamma^2}{64}\,\frac{\omega_1^2\omega_2^2 h_0^2}{|\alpha_1(\omega_s-\omega_2-\Delta)\alpha_2(\omega_s-\omega_2) - \tfrac{1}{16}\omega_1\omega_2 h_0^2|^2},$$
$$k_{22}^2(\omega_s) = \frac{\omega_2^2}{4}\,\frac{|\alpha_1(\omega_s-\omega_2-\Delta)|^2}{|\alpha_1(\omega_s-\omega_2-\Delta)\alpha_2(\omega_s-\omega_2) - \tfrac{1}{16}\omega_1\omega_2 h_0^2|^2}. \tag{10.113}$$

The above results are valid even if the amplitudes and phases of the signals are not exactly constant, provided they do not change much during times of order T_1, T_2.

6.2. Next we consider the case where the voltage $E_1(t)$ is a random process $\xi(t)$ with spectral density $S[\xi;\omega]$, instead of a pure sine wave. The fluctuating voltage $\xi(t)$ may be applied to the circuit along with a sine wave, or it may arise because of thermal noise in the ohmic resistance R_1. A third possibility is that $\xi(t)$ is due to fluctuations of the current flowing through the variable capacitance $C(t)$, e.g., of the shot-noise type. The situation is the same in the second branch. All of these effects destroy the stability of the oscillatory regime, and lead to deviations from the stable values (10.110). It follows from (10.103) that these fluctuations are described by the equations

$$\dot{z}_1 + \delta_1 z_1 - \frac{\omega_1\gamma}{4} h e^{i\Delta t} z_2^* = \omega_1 \xi'(t),$$
$$\dot{z}_2^* + \delta_2 z_2^* - \frac{\omega_2}{4\gamma} h^* e^{-i\Delta t} z_1 = 0, \tag{10.114}$$

where $\xi'(t) = -ie^{-i\omega_1 t}\xi(t)$ is a zero-mean random function with correlation function

$$\langle \xi' \xi'^*_\tau \rangle = e^{i\omega_1 \tau} \langle \xi \xi_\tau \rangle$$

and spectral density

$$S[\xi'; \omega] = 2\int \langle \xi' \xi'^*_\tau \rangle e^{i\omega\tau} d\tau = S[\xi; \omega + \omega_1]. \qquad (10.115)$$

Substituting the spectral expansions (cf. p. I.28)

$$z_1(t) = \frac{1}{\sqrt{2\pi}} \int e^{i\omega t} z_1(\omega)\, d\omega,$$

$$z_2(t) = \frac{1}{\sqrt{2\pi}} \int e^{i\omega t} z_2(\omega)\, d\omega,$$

$$\xi'(t) = \frac{1}{\sqrt{2\pi}} \int e^{i\omega t} \xi'(\omega)\, d\omega$$

of the processes z_1, z_2, ξ' into (10.114), and equating spectral components of the same frequency in the resulting equations, we find that

$$(i\omega + \delta_1) z_1(\omega) - \frac{\omega_1 \gamma}{4} h z_2^*(\Delta - \omega) = \omega_1 \xi'(\omega),$$

$$[-i(\Delta - \omega) + \delta_2] z_2^*(\Delta - \omega) - \frac{\omega_2}{4\gamma} h^* z_1(\omega) = 0.$$

This system of equations has the solution

$$z_1(\omega) = \omega_1 \frac{\alpha_2^*(\Delta - \omega)}{\alpha_1(\omega)\alpha_2^*(\Delta - \omega) - \frac{1}{16}\omega_1\omega_2 h_0^2} \xi'(\omega),$$

$$z_2(\omega) = \frac{1}{4\gamma} \frac{\omega_1\omega_2 h}{\alpha_1^*(\Delta - \omega)\alpha_2(\omega) - \frac{1}{16}\omega_1\omega_2 h_0^2} \xi'^*(\Delta - \omega),$$

analogous to (10.110), where

$$\alpha_1(\Omega) = \delta_1 + i\Omega, \quad \alpha_2(\Omega) = \delta_2 + i\Omega.$$

Therefore the corresponding spectral densities are

$$S[z_1;\omega] = \omega_1^2 \frac{\delta_2^2 + (\Delta-\omega)^2}{|\alpha_1(\omega)\alpha_2^*(\Delta-\omega) - \frac{1}{16}\omega_1\omega_2 h_0^2|^2} S[\xi';\omega],$$

$$S[z_2;\omega] = \frac{1}{16\gamma^2} \frac{\omega_1^2\omega_2^2 h_0^2}{|\alpha_1^*(\Delta-\omega)\alpha_2(\omega) - \frac{1}{16}\omega_1\omega_2 h_0^2|^2} S[\xi';\Delta-\omega].$$

The spectral density $S[x_1;\omega]$ of the process $x_1 = \mathrm{Re}\{z_1 e^{-i\omega t}\}$ can be found by using formula (I.7.13). In fact, taking account of (10.115) as well, we can express $S[x_1;\omega]$ in terms of $S[\xi;\omega]$ and the amplification factor (10.111):

$$S[x_1;\omega] = k_{11}^2(\omega) S[\xi;\omega]. \tag{10.116}$$

Similarly, we obtain

$$S[x_2;\omega] = k_{21}^2(\nu-\omega) S[\xi;\nu-\omega]. \tag{10.117}$$

The fluctuations caused by a random e.m.f. $\xi(t) = E_2(t)$ in the second branch are investigated in the same way, and the corresponding spectral densities are given by the formulas

$$\begin{aligned} S[x_1;\omega] &= k_{12}^2(\nu-\omega) S[\xi;\nu-\omega], \\ S[x_2;\omega] &= k_{22}^2(\omega) S[\xi;\omega], \end{aligned} \tag{10.118}$$

analogous to (10.116) and (10.117). To obtain expressions for the spectral densities of the branch currents themselves, we note that $I_1 = C_1\dot{x}_1$ (say), and hence

$$S[I_1;\omega] = \omega^2 C_1^2 S[x_1;\omega].$$

6.3. Besides additive fluctuations appearing in E_1 and E_2, we can also have parametric fluctuations, i.e., a random component $\zeta(t)$ in the function

$$H(t) = h_0 \sin(\nu t + \psi) + \zeta(t).$$

As already noted, the oscillatory process is affected mainly by the spectral components of the fluctuations $\zeta(t)$ in the four regions near the frequencies 0, $2\omega_1$, $|\omega_1 - \omega_2|$ and $\omega_1 + \omega_2$. If the fluctuations are not too intense, these regions can be considered separately. We can then add the spectral densities of the fluctuations contributed by the various regions and also by the voltages E_1 and E_2.

Confining our attention to fluctuations $\zeta(t)$ in the frequency band $|\omega-\omega_1-\omega_2| \ll \omega_{1,2}$, we can drop the terms $\zeta(t), \zeta(t)\, e^{-2i\omega_1 t}$ and $\zeta(t)e^{-i(\omega_2-\omega_1)t}$ in (10.103). This gives

$$\begin{aligned} \dot{z}_1 + \delta_1 z_1 - \frac{\omega_1\gamma}{4} h e^{i\Delta t} z_2^* &= \frac{\omega_1}{2} E e^{i\Delta_1 t} + \frac{\omega_1\gamma}{2} \zeta'(t) z_2^*, \\ \dot{z}_2^* + \delta_2 z_2^* - \frac{\omega_2}{4\gamma} h^* e^{-i\Delta t} z_1 &= \frac{\omega_2}{2\gamma} \zeta'^*(t) z_1, \end{aligned} \tag{10.119}$$

where $\zeta'(t)=ie^{-i(\omega_1+\omega_2)t}\, \zeta(t)$ is a zero-mean function with correlation function and spectral density

$$\begin{aligned} \langle \zeta'\zeta'^*_\tau \rangle &= e^{i(\omega_1+\omega_2)\tau} \langle \zeta\zeta_\tau \rangle, \\ S[\zeta'; \omega] &= S[\zeta; \omega + \omega_1 + \omega_2]. \end{aligned} \tag{10.120}$$

The influence of the parametric fluctuations, described by the terms $\langle \zeta' z_2^* \rangle$ and $\langle \zeta'^* z_1 \rangle$ in (10.119), appears in two ways. In the first place, the fluctuations change the mean values $\langle z_1 \rangle$ and $\langle z_2 \rangle$ of the complex amplitudes, and secondly, they cause extra fluctuational scatter about these mean values. To study the first effect, we assume that the spectral density is approximately constant in the narrow frequency band

$$|\omega - \omega_1 - \omega_2| \sim \frac{1}{T_{1,2}}.$$

This allows us to regard the correlation time τ_{cor} of the random process $\zeta'(t)$ in (10.119) as much smaller than the relaxation time $T_{1,2}$ characterizing the time it takes the amplitudes to change appreciably:

$$\tau_{cor} \ll T_{1,2}. \tag{10.121}$$

Averaging the equations (10.119), we obtain

$$\begin{aligned} \langle \dot{z}_1 \rangle + \delta_1 \langle z_1 \rangle - \frac{\omega_1\gamma}{4} h e^{i\Delta t} \langle z_2^* \rangle &= \frac{\omega_1}{2} E e^{i\Delta_1 t} + \frac{\omega_1\gamma}{2} \langle \zeta' z_2^* \rangle, \\ \langle \dot{z}_2^* \rangle + \delta_2 \langle z_2^* \rangle - \frac{\omega_2}{4\gamma} h^* e^{-i\Delta t} \langle z_1 \rangle &= \frac{\omega_2}{2\gamma} \langle \zeta'^* z_1 \rangle. \end{aligned} \tag{10.122}$$

To calculate the averages $\langle \zeta' z_2^* \rangle$ and $\langle \zeta'^* z_1 \rangle$, we use the method given on pp. 110–113, based on the condition (10.121). Let Δ_0 be a time shift

such that $\tau_{cor} \ll \Delta_0 \ll T_{1,2}$. Then there is negligible dependence between $\zeta'(t)$ and $z_2^*(t-\Delta_0)$, since $\tau_{cor} \ll \Delta_0$, and hence

$$\langle \zeta'(t) z_2^*(t) \rangle = \langle \zeta'(t) [z_2^*(t) - z_2^*(t-\Delta_0)] \rangle . \qquad (10.123)$$

Moreover, during the time Δ_0 the amplitudes do not manage to change appreciably. Therefore, integrating the second of the equations (10.119), we obtain

$$z_2^*(t) - z_2^*(t-\Delta_0) \approx -\delta_2 z_2^*(t-\Delta_0)\Delta_0 + \frac{\omega_2}{4\gamma} h^* e^{-i\Delta t} z_1(t-\Delta_0)\Delta_0 + \frac{\omega_2}{2\gamma} z_1(t-\Delta_0) \int_{t-\Delta_0}^{t} \zeta'^*(\sigma)\, d\sigma . \qquad (10.124)$$

Substitution of (10.124) into (10.123) gives

$$\langle \zeta' z_2^* \rangle = \frac{\omega_2}{2\gamma} \langle z_1 \rangle \int_{-\infty}^{0} \langle \zeta' \zeta'^*_\tau \rangle\, d\tau = \frac{\omega_2}{4\gamma} \langle z_1 \rangle \kappa ,$$

where

$$\kappa = 2 \int_{-\infty}^{0} \langle \zeta' \zeta'^*_\tau \rangle\, d\tau = 2 \int_{-\infty}^{0} \langle \zeta \zeta_\tau \rangle e^{i(\omega_1+\omega_2)\tau} d\tau$$

$$= \tfrac{1}{2} S[\zeta; \omega_1 + \omega_2] + 2i \int_{-\infty}^{0} \langle \zeta \zeta_\tau \rangle \sin[(\omega_1+\omega_2)\tau]\, d\tau .$$

In the same way, we find that

$$\langle \zeta'^* z_1 \rangle = \frac{\omega_1 \gamma}{4} \langle z_2^* \rangle \kappa^* .$$

As a result, the equations (10.122) take the form

$$\begin{aligned} &\langle \dot z_1 \rangle + \left(\delta_1 - \frac{\omega_1\omega_2}{8}\kappa\right)\langle z_1 \rangle - \frac{\omega_1\gamma}{4} h e^{i\Delta t} \langle z_2^* \rangle = \frac{\omega_1}{2} E e^{i\Delta_1 t} , \\ &\langle \dot z_2^* \rangle + \left(\delta_2 - \frac{\omega_1\omega_2}{8}\kappa^*\right)\langle z_2^* \rangle - \frac{\omega_2}{4\gamma} h^* e^{-i\Delta t} \langle z_1 \rangle = 0 . \end{aligned} \qquad (10.125)$$

These equations differ from (10.106) only by having

$$\tilde{\delta}_1 = \delta_1 - \frac{\omega_1\omega_2}{8}\kappa, \quad \tilde{\delta}_2 = \delta_2 - \frac{\omega_1\omega_2}{8}\kappa^*$$

instead of δ_1 and δ_2. Taking account of the real part of these corrections, we conclude that the presence of parametric fluctuations decreases the damping in each branch by an amount

$$\frac{\omega_1\omega_2}{16} S[\zeta; \omega_1 + \omega_2].$$

The solution of the system (10.125) is the same as that of (10.106), provided we replace $\alpha_1(\Omega)$ and $\alpha_2(\Omega)$ in formulas (10.107) and (10.110)–(10.113) by

$$\begin{aligned} \tilde{\alpha}_1(\Omega) &= \delta_1 - \frac{\omega_1\omega_2}{8}\kappa + i\Omega, \\ \tilde{\alpha}_2(\Omega) &= \delta_2 - \frac{\omega_1\omega_2}{8}\kappa^* + i\Omega. \end{aligned} \tag{10.126}$$

Then the stationary mean amplitudes corresponding to (10.110) become

$$\begin{aligned} \langle z_1' \rangle = \langle z_1 \rangle e^{-i\Delta_1 t} &= \frac{\omega_1}{2} \frac{\tilde{\alpha}_2(-\Delta_2)}{\tilde{\alpha}_1(\Delta_1)\tilde{\alpha}_2(-\Delta_2) - \frac{1}{16}\omega_1\omega_2 h_0^2} E, \\ \langle z_2' \rangle = \langle z_2 \rangle e^{-i\Delta_2 t} &= \frac{1}{8\gamma} \frac{\omega_1\omega_2 h}{\tilde{\alpha}_1(-\Delta_1)\tilde{\alpha}_2(\Delta_2) - \frac{1}{16}\omega_1\omega_2 h_0^2} E^*. \end{aligned} \tag{10.127}$$

The new values of amplification factors (10.111)–(10.113), with α_1, α_2 replaced by $\tilde{\alpha}_1, \tilde{\alpha}_2$, will be denoted by $\tilde{k}_{11}, \tilde{k}_{21}, \tilde{k}_{12}, \tilde{k}_{22}$.

We now turn to the second effect, i.e., the extra scatter about the mean values $\langle z_1 \rangle$ and $\langle z_2 \rangle$ due to the parametric fluctuations $\zeta(t)$. Introducing the deviations

$$\delta z_1 = z_1 - \langle z_1 \rangle, \quad \delta z_2 = z_2 - \langle z_2 \rangle, \tag{10.128}$$

we deduce from (10.119) and (10.125) that

$$\begin{aligned} \delta\dot{z}_1 + \tilde{\delta}_1\delta z_1 - \frac{\omega_1\gamma}{4} h\, e^{i\Delta t}\delta z_2^* &= \frac{\omega_1\gamma}{2}\langle z_2^* \rangle \zeta', \\ \delta\dot{z}_2^* + \tilde{\delta}_2\delta z_2 - \frac{\omega_2 h^*}{4\gamma} e^{-i\Delta t}\delta z_1 &= \frac{\omega_2}{2\gamma}\langle z_1 \rangle \zeta'^*, \end{aligned} \tag{10.129}$$

if the fluctuations are small. These equations resemble (10.114), but the attenuations are different and both equations contain fluctuational terms. The correspondence between (10.129) and (10.114) becomes more complete if we introduce the random processes

$$\begin{aligned}\xi_1(t) &= \frac{\gamma}{2}\langle z_2^*\rangle \zeta' e^{i\omega_1 t} = \frac{i\gamma}{2}\langle z_2'^*\rangle \zeta e^{-i(\nu-\omega_s)t},\\ \xi_2(t) &= \frac{1}{2\gamma}\langle z_1^*\rangle \zeta' e^{i\omega_2 t} = \frac{i}{2\gamma}\langle z_1'^*\rangle \zeta e^{-i\omega_s t}.\end{aligned} \tag{10.130}$$

with spectral densities

$$\begin{aligned}S[\xi_1;\omega] &= \frac{\gamma^2}{4}|\langle z_2\rangle|^2 S[\zeta;\omega+\nu-\omega_s],\\ S[\xi_2;\omega] &= \frac{1}{4\gamma^2}|\langle z_1\rangle|^2 S[\zeta;\omega+\omega_s].\end{aligned} \tag{10.131}$$

The stationary mean amplitudes $\langle z_1\rangle$ and $\langle z_2\rangle$ appearing in (10.131) correspond to the stationary regime are given by (10.127). Moreover, the random processes $\xi_1(t)$ and $\xi_2(t)$ can be regarded as uncorrelated, since their cross-correlation function is proportional to the expression

$$\langle \zeta\zeta_\tau\rangle e^{-i(\nu-2\omega_s)t+i\omega_s\tau}$$

which vanishes when averaged with respect to t. It follows from (10.131) and (10.116)–(10.118) that .

$$\begin{aligned}S[x_1;\omega] &= k_{11}^2(\omega)\frac{\gamma^2}{4}|\langle z_2\rangle|^2 S[\zeta;\omega+\nu-\omega_s]\\ &\quad + k_{12}^2(\nu-\omega)\frac{1}{4\gamma^2}|\langle z_1\rangle|^2 S[\zeta;\nu+\omega_s-\omega],\\ S[x_2;\omega] &= k_{21}^2(\nu-\omega)\frac{\gamma^2}{4}|\langle z_2\rangle|^2 S[\zeta;2\nu-\omega_s-\omega|\\ &\quad + k_{22}^2(\omega)\frac{1}{4\gamma^2}|\langle z_1\rangle|^2 S[\zeta;\omega+\omega_s].\end{aligned}$$

or

$$S[x_1;\omega] = \frac{\gamma^2 E^2}{4} k_{21}^2(\omega_s) k_{11}^2(\omega) S[\zeta;\omega+\nu-\omega_s] + \frac{E^2}{4\gamma^2} k_{11}^2(\omega_s) k_{12}^2(\nu-\omega) S[\zeta;\nu+\omega_s-\omega],$$

$$S[x_2;\omega] = \frac{\gamma^2 E^2}{4} k_{21}^2(\omega_s) k_{21}^2(\nu-\omega) S[\zeta;2\nu-\omega_s-\omega] + \frac{E^2}{4\gamma^2} k_{11}^2(\omega_s) k_{22}^2(\omega) S[\zeta;\omega+\omega_s] \qquad (10.132)$$

in terms of the various amplification factors [cf. (10.110)–(10.113)]. According to these formulas, the effect of the parametric fluctuations $\zeta(t)$ on the processes x_1 and x_2 is proportional to the square of the amplitude E of the signal being amplified.

The first of the equations (10.132) shows that the spectral density of the process x_1 near the frequency ω_s is proportional to the spectral density of the parametric fluctuations near the frequency ν. If the spectral density $S[\zeta;\omega]$ is approximately constant for both $|\omega-\nu| \sim \delta_1$, δ_2 and $|\omega-\nu| \sim \Delta_1, \Delta_2$, then the spectral densities on the right in (10.132) can all be replaced by $S[\zeta;\nu]$. Integrating the first of the equations (10.132) and taking account of (I.2.15), we find that the variance of the amplified signal is given by

$$\frac{\sigma^2(x_1)}{E^2} = \frac{1}{8\pi} S[\zeta;\nu] \left\{ \gamma^2 k_{21}^2(\omega_s) \int_0^\infty k_{11}(\omega)\, d\omega + \frac{1}{\gamma^2} k_{11}^2(\omega_s) \int_0^\infty k_{12}^2(\omega)\, d\omega \right\}$$

Therefore the parametric fluctuations give a constant signal-to-noise ratio, regardless of the strength of the signal:

$$\frac{k_{11}(\omega_s) E}{\sigma(x_1)} = 2\sqrt{2\pi\gamma} \left\{ S[\zeta;\nu] \left(\int_0^\infty k_{12}^2(\omega)\, d\omega + \gamma^4 \frac{k_{21}^2(\omega_s)}{k_{11}^2(\omega_s)} \int_0^\infty k_{11}^2(\omega)\, d\omega \right) \right\}^{-1/2}.$$

The presence of the neglected terms in the right-hand side of (10.103) causes other frequency regions in the spectral density $S[\zeta;\omega]$ to affect the fluctuational scatter of the processes x_1 and x_2. For example, the term $Hz_2 e^{i(\omega_2-\omega_1)t}$ leads to the appearance of fluctuational components with intensity proportional to $S[\zeta;\omega_1-\omega_2]$, which satisfy formulas analogous to those derived above.

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Here we cite some important papers on the topics covered in Parts 1 and 2. For general references on noise theory, see the Bibliography on p. 287 of Volume I.

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Addendum

Since the publication of Volume I, the collection *Non-Linear Transformations of Stochastic Processes*, edited by P. I. Kuznetsov, R. L. Stratonovich and V. I. Tikhonov has appeared in English translation. The papers cited on pp. I.47, I.62, I.143, I.149 and I.205 (bis) all appear in this collection, on pp. 59, 77, 405, 101, 3 and 29, respectively.

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